Conjugation in Semigroups

João Araújo*†
Universidade Aberta, R. Escola Politécnica, 147
1269-001 Lisboa, Portugal
&
Centro de Álgebra da Universidade de Lisboa
1649-003 Lisboa, Portugal, mjoao@lmc.fc.ul.pt

Janusz Konieczny‡
Department of Mathematics, University of Mary Washington
Fredericksburg, VA 22401, jkoniecz@umw.edu

António Malheiro*
Departamento de Matemática, Faculdade de Ciências e Tecnologia,
Universidade Nova de Lisboa, 2829-516 Caparica, Portugall
&
Centro de Álgebra da Universidade de Lisboa
1649-003 Lisboa, Portugal, malheiro@cii.fc.ul.pt

Abstract

The action of any group on itself by conjugation and the corresponding conjugacy relation play an important role in group theory. There have been several attempts to extend the notion of conjugacy to semigroups. In this paper, we present a new definition of conjugacy that can be applied to an arbitrary semigroup and it does not reduce to the universal relation in semigroups with a zero. We compare the new notion of conjugacy with some existing definitions and study the decidability of the conjugacy problem for certain classes of finitely presented monoids. We characterize the conjugacy in various semigroups of transformations on a set, and count the number of conjugacy classes in these semigroups in the case when the set is infinite.

The paper ends with a number of problems for experts in combinatorics, symbolic dynamics, set theory, semigroups, and matrix theory.

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1 Introduction

Let G be a group. For elements $a, b \in G$, we say that a is *conjugate* to b if there exists $g \in G$ such that $b = g^{-1}ag$. It is clear that this relation is an equivalence on G and that a is conjugate to b if and only if there exists $g \in G$ such that $ag = gb$. Using the latter formulation, one may try to extend the notion of conjugacy to semigroups in the following way: for all elements a and

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b in a semigroup S, we say that a is *left conjugate* to b, and write $a \sim_l b$, if $ag = gb$ for some $g \in S^1$, where S^1 is S with an identity adjoined [48, 55, 56]. That is,

$$a \sim_l b \Leftrightarrow \exists_{g \in S^1}\ ag = gb. \tag{1.1}$$

(We will write "$\sim$" with various subscripts for possible definitions of conjugacy in semigroups. The subscript in $\sim_l$ comes from the name "left conjugate.") In a general semigroup S, the relation $\sim_l$ is reflexive and transitive, but not symmetric.

The relation $\sim_l$ is an equivalence in any free semigroup. Lallement [33] has defined the conjugate elements of a free semigroup S as those related by $\sim_l$ and showed that $\sim_l$ is equal to the following equivalence on the free semigroup S:

$$a \sim_p b \Leftrightarrow \exists_{u,v \in S^1}\ a = uv \text{ and } b = vu. \tag{1.2}$$

In a general semigroup S, the relation $\sim_p$ is reflexive and symmetric, but not transitive. If $a \sim_p b$ in a general semigroup, we say that a and b are *primarily conjugate* [32] (hence the subscript in $\sim_p$). Kudryavtseva and Mazorchuk [31, 32] have defined the transitive closure $\sim_p^*$ of $\sim_p$ as a conjugacy relation in a general semigroup. (See also [25].)

Otto [48] has studied the relations $\sim_l$ and $\sim_p$ in the monoids S presented by finite Thue systems, and introduced a new definition of conjugate elements in such an S:

$$a \sim_o b \Leftrightarrow \exists_{g,h \in S^1}\ ag = gb \text{ and } bh = ha. \tag{1.3}$$

(Since S is a monoid, $S^1 = S$. However, we wanted to write the definition of $\sim_o$ so that it would apply to any semigroup.) The relation $\sim_o$ is an equivalence on an arbitrary semigroup S, and so it provides another possible definition of conjugacy in a general semigroup. However, this definition is not useful for semigroups S with zero since for every such S, we have $\sim_o = S \times S$.

The aim of this paper is to introduce a new definition of conjugacy in an arbitrary semigroup, avoiding the problems of the notions listed above. (That is, $\sim_l$ is not symmetric; both $\sim_l$ and $\sim_o$ reduce to the universal relation in semigroups with a zero; and $\sim_p$ is not transitive and so it requires taking the transitive closure.) Our conjugacy will be an equivalence relation $\sim_c$ on any semigroup S, it will not reduce to the universal relation even when S has a zero, and it will be such that $\sim_c \subseteq \sim_o \subseteq \sim_l$ in every semigroup S, $\sim_c = \sim_o$ if S is a semigroup without zero, and $\sim_c = \sim_l = \sim_p = \sim_o$ if S is a group or a free semigroup.

In Section 2 we introduce the new notion of conjugacy and prove some basic results. Section 3 is dedicated to the decidability results for this new notion. The following four sections are devoted to the study of $\sim_c$ in several transformation semigroups on a finite or infinite set X. The tools we use in this study are the characterization of $\sim_c$ in transformation semigroups in terms of certain partial homomorphisms of directed graphs (Section 4) and the concept of a connected partial transformation (Section 5). Conjugacy classes in the full transformation monoid $T(X)$ are characterized (for any X) and counted (for an infinite X) in Section 6; conjugacy in the partial transformation monoid $P(X)$ is treated in Section 7; and Section 8 deals with the monoid $\Gamma(X)$ of full injective transformations. The paper ends with a number of problems for experts in combinatorics, symbolic dynamics, set theory, semigroups, and matrix theory (Section 9).

2 Definition of Conjugacy

We briefly describe the motivation of our new concept of conjugacy. The starting point was the relation $\sim_o$ introduced by Otto [48]. As we have already pointed out, the relation $\sim_o$ is the universal relation $S \times S$ if a semigroup S has a zero. Our goal has been to retain Otto's concept for semigroups without zero but modify his definition in such a way that the resulting conjugacy would not reduce to triviality for semigroups with zero.

To find a suitable definition, we considered the semigroup $P(X)$ of partial transformations on X, that is, the set of all mappings whose domain and image are included in X, with function

composition as multiplication. This semigroup has the zero, namely the transformation whose domain is empty. Let $\alpha, \beta \in P(X)$. Then $\alpha \sim_o \beta$ if and only if $\alpha\phi = \phi\beta$ and $\beta\psi = \psi\alpha$ for some $\phi, \psi \in P(X)$. (We write functions on the right and compose from left to right.) Of course, the last two equalities hold for $\phi = \psi = 0$. We could insist that ϕ and ψ should not be 0 but this would not solve the problem since then the resulting relation would not be transitive.

The solution is this. In the composition $\alpha\phi$, it only matters how ϕ is defined on the elements of $\mathrm{im}(\alpha)$ (the image of α). We insist that ϕ be defined for *all* elements of $\mathrm{im}(\alpha)$, that is, that $\mathrm{im}(\alpha) \subseteq \mathrm{dom}(\phi)$. With the requirement that the transformations ϕ and ψ come from the sets $\{\phi \in P(X) : \mathrm{dom}(\phi) \subseteq \mathrm{im}(\alpha)\}$ and $\{\psi \in P(X) : \mathrm{dom}(\psi) \subseteq \mathrm{im}(\beta)\}$, the relation is an equivalence. Moreover, we will prove that for $\alpha \neq 0$, $\mathrm{dom}(\phi) \subseteq \mathrm{im}(\alpha)$ if and only if $(\gamma\alpha)\phi \neq 0$ for every $\gamma\alpha \in P(X)\alpha - \{0\}$, where $P(X)\alpha - \{0\}$ is the left principal ideal generated by α with 0 removed. Therefore, the requirement that ϕ and ψ have "large" domains can be expressed in an abstract semigroup. These considerations motivate the definition below.

Let S be a semigroup with zero. For $a \in S$ with $a \neq 0$, consider $S^1a - \{0\}$, the left principal ideal generated by a with zero removed. We will denote by $\mathbb{P}(a)$ the set of all elements $g \in S$ such that $(ma)g \neq 0$ for all $ma \in S^1a - \{0\}$. We define $\mathbb{P}(0)$ to be $\{0\}$. If S has no zero, we agree that $\mathbb{P}(a) = S$ for every $a \in S$. We will write $\mathbb{P}^1(a)$ for $\mathbb{P}(a) \cup \{1\}$, where 1 is the identity in S^1.

Definition 2.1. Let S be a semigroup. For $a, b \in S$, we say that a is *conjugate* to b, written $a \sim_c b$, if there are $g \in \mathbb{P}^1(a)$ and $h \in \mathbb{P}^1(b)$ such that $ag = gb$ and $bh = hb$. That is,

$$a \sim_c b \Leftrightarrow \exists_{g \in \mathbb{P}^1(a)} \exists_{h \in \mathbb{P}^1(b)}\ ag = gb \text{ and } bh = ha. \tag{2.1}$$

The relation $\sim_c$ on a semigroup S will be called the *conjugacy* on S, which is justified by the following theorem.

Theorem 2.2. *Let S be a semigroup. Then:*

(1) *The relation $\sim_c$ is an equivalence relation on S.*

(2) *If $\sim_l$, $\sim_p$, and $\sim_o$ are relations on S defined by* (1.1), (1.2), and (1.3), *respectively, then:*

  (a) $\sim_c \subseteq \sim_o \subseteq \sim_l$ *and* $\sim_p \subseteq \sim_o \subseteq \sim_l$,

  (b) *If S is a semigroup without zero, then $\sim_c = \sim_o$, and*

  (c) *If S is a group or a free semigroup, then $\sim_c = \sim_l = \sim_p = \sim_o$.*

Proof. It is clear that $\sim_c$ is reflexive and symmetric. Suppose $a \sim_c b$ and $b \sim_c c$. Then there are $g_1 \in \mathbb{P}^1(a)$ and $g_2 \in \mathbb{P}^1(b)$ such that $ag_1 = g_1b$ and $bg_2 = g_2c$. Thus $a(g_1g_2) = (ag_1)g_2 = (g_1b)g_2 = g_1(bg_2) = g_1(g_2c) = (g_1g_2)c$. Let $ma \in S^1a - \{0\}$. Since $g_1 \in \mathbb{P}^1(a)$, we have $(mg_1)b = m(ag_1) = (ma)g_1 \neq 0$. Thus $(mg_1)b \in S^1b - \{0\}$, and so, since $g_2 \in \mathbb{P}^1(b)$, we have $(ma)(g_1g_2) = m(ag_1)g_2 = m(g_1b)g_2 = ((mg_1)b)g_2 \neq 0$. Hence $g_1g_2 \in \mathbb{P}^1(a)$. Similarly, there is $h \in \mathbb{P}^1(c)$ such that $ch = ha$. Hence $a \sim_c c$, and so $\sim_c$ is transitive. We have proved (1).

Statements 2(a) and 2(b) follow immediately from the definitions of $\sim_l$, $\sim_p$, $\sim_o$, and $\sim_c$. (Note that $\sim_p \subseteq \sim_o$ since if $a = uv$ and $b = vu$, then $au = ub$ and $bv = va$.) Statement 2(c) is clearly true if S is a group. Let S be a free semigroup. Then $\sim_l = \sim_p$ by [33, Corollary 5.2]. Thus, by 2(a) and 2(b), $\sim_c = \sim_o \subseteq \sim_l = \sim_p \subseteq \sim_o$, which implies $\sim_c = \sim_o = \sim_l = \sim_p$. ∎

For an element a of a semigroup S, the equivalence class of a with respect to $\sim_c$ will be called the *conjugacy* class of a and denoted $[a]_c$.

Let S be a semigroup with 0. In contrast with the fact that $\sim_o = S \times S$, the conjugacy class of 0 with respect to $\sim_c$ is $\{0\}$, so we always have $\sim_c \neq S \times S$ unless $S = \{0\}$. Indeed, suppose $a \sim_c 0$. Then $ag = g0 = 0$ for some $g \in \mathbb{P}^1(a)$. If $a \neq 0$, then $ag \neq 0$ (since $a \in S^1a - \{0\}$). But $ag = 0$, and so it follows that $a = 0$. Hence $[0]_c = \{0\}$.

For a set A, we denote by Δ_A (or Δ if A is understood) the identity relation on A, that is $\Delta_A = \{(a,a) : a \in A\}$. Recall that in any group G, the relation $\sim_c$ is the usual group conjugacy, that is $a \sim_c b$ if and only if $g^{-1}ag = b$ for some $g \in G$. It follows that in any group G, we have $\sim_c = \Delta$ if and only if G is commutative. This result extends to semigroups as follows.

Theorem 2.3. *Let S be a finite semigroup without zero. Then $\sim_c = \Delta$ if and only if S is a commutative group.*

Proof. It is clear that if S is a commutative group, then $\sim_c = \Delta$. Conversely, suppose that $\sim_c = \Delta$. Recall that in a semigroup S without zero, $\mathbb{P}(a) = S$ for every $a \in S$, and so $a \sim_c b$ if and only if $ag = gb$ and $bh = hb$ for some $g, h \in S^1$. Let $a, b \in S$. Since $(ab)a = a(ba)$ and $(ba)b = b(ab)$, we have $(ab) \sim_c (ba)$, and hence $ab = ba$. We have proved that S is commutative. Let $a, b, c \in S$ be such that $ac = bc$. Since S is commutative, $ac = bc$ implies $a \sim_c b$, which in turn implies $a = b$ (since $\sim_c = \Delta$). We have proved that S is right cancellative. As S is commutative, S is also left cancellative, and so it is a group [26, Exercise 5, p. 61]. $\square$

Theorem 2.3 is not true for semigroups with zero. For example, let $S = \{a, 0\}$ be a 2-element semigroup with zero (with $aa = a$ or $aa = 0$). Then S is not a group but we already know that $[0]_c = \{0\}$, so $\sim_c = \Delta$.

3 Decidability Results for the c-conjugacy

In this section, we present some results concerning the conjugacy relations from Sections 1 and 2 in monoids defined by rewriting systems, with the focus on the decidability of the various conjugacy problems. These considerations will amplify the relations between the new notion of conjugacy, $\sim_c$, and the notions previously considered in the literature, $\sim_l$, $\sim_p$, and $\sim_o$.

In 1911, Max Dehn [19] formulated three fundamental decision problems concerning finitely presented groups: the word problem, the conjugacy problem, and the isomorphism problem. By using finite monoid presentations, these questions have been considered in the more general setting of semigroups. (For some classic results, see [11, 40, 45, 49]. For some results concerning various notions of conjugacy in semigroups, see [15, 16, 41, 42, 43, 44, 48, 50, 55, 56].)

We will introduce some basic notation specific to this section. Let Σ be a non-empty set, called an *alphabet*. We denote by Σ^* the set of finite strings (called words) of elements of Σ, including the empty word 1. For $w \in \Sigma^*$ and $a \in \Sigma$, we denote by $|w|$ the length of the word w and by $|w|_a$ the number of occurrences of a in w. For example, if $\Sigma = \{a, b, c\}$ and $w = aabba \in \Sigma^*$, then $|w| = 5$, $|w|_a = 3$, and $|w|_c = 0$.

Any subset R of $\Sigma^* \times \Sigma^*$ is called a *rewriting system* (or a *Thue system*) on Σ. (We will also refer to the pair $(\Sigma; R)$ as a rewriting system.) An element (x, y) of R is called a *rewriting rule*. If $(x, y) \in R$ and $u, v \in \Sigma^*$, we say that uxv *reduces* to uyv and we write $uxv \underset{R}{\longrightarrow} uyv$. We denote by $\overset{*}{\underset{R}{\longrightarrow}}$ the reflexive and transitive closure of $\underset{R}{\longrightarrow}$.

A rewriting system R on Σ is *special* if every element of R is of the form $(x, 1)$ with $x \neq 1$; it is *monadic* if every element of R is of the form (x, y) with $y \in \Sigma \cup \{1\}$ and $|x| > |y|$; it is *length reducing* if $|x| > |y|$ for all $(x, y) \in R$; it is *Noetherian* if there is no infinite sequence $w_1, w_2, w_3, \ldots$ of words in Σ^* such that $w_1 \underset{R}{\longrightarrow} w_2 \underset{R}{\longrightarrow} w_3 \underset{R}{\longrightarrow} \cdots$; it is *confluent* if for all $u, v, w \in \Sigma^*$, if $u \overset{*}{\underset{R}{\longrightarrow}} v$ and $u \overset{*}{\underset{R}{\longrightarrow}} w$, then there exists $z \in \Sigma^*$ such that $v \overset{*}{\underset{R}{\longrightarrow}} z$ and $w \overset{*}{\underset{R}{\longrightarrow}} z$; and R is *complete* if it is both Noetherian and confluent. Note that if R is special or monadic, then it is length reducing, and if R is length reducing, then it is Noetherian.

Every rewriting system R on Σ defines a monoid. The set Σ^* with concatenation of words as multiplication is a monoid, called the *free monoid* on Σ. Denote by $\overset{*}{\underset{R}{\longleftrightarrow}}$ the smallest congruence on Σ^* containing R (called the *Thue congruence*). We denote by $M(\Sigma; R)$ the quotient monoid

$\Sigma^*/\overset{*}{\underset{R}{\longleftrightarrow}}$. The elements of $M(\Sigma;R)$ are the congruence classes $[u]_M = \{w : w \overset{*}{\underset{R}{\longleftrightarrow}} u\}$, where $u \in \Sigma^*$. Suppose M is any monoid such that $M \cong M(\Sigma;R)$ (that is, M is isomorphic to $M(\Sigma;R)$). Then the rewriting system $(\Sigma;R)$ is a *presentation* of M with generators Σ and defining relations R, and we say that M is defined by $(\Sigma;R)$ or simply by R. A rewriting system $(\Sigma;R)$ is said to be *finite* if both Σ and R are finite. A monoid M defined by a finite presentation is called *finitely presented*. We refer the reader to [12] for more details.

Definition 3.1. Let M be a monoid with a finite presentation $(\Sigma;R)$ and let $\sim_i$ be one of the conjugacy relations introduced Section 1 ($i \in \{l,p,o,c\}$). We say that the i-conjugacy problem for M is *decidable* if there is an algorithm that given any pair (u,v) of words in Σ^*, returns YES if $[u]_M \sim_i [v]_M$ and NO otherwise. If such an algorithm does not exists, we say that the i-conjugacy problem for M is *undecidable*.

The conjugacy problem for finitely presented groups is undecidable [45] (that is, there is a finitely presented group G such that the conjugacy problem for G is undecidable). Suppose G is a finitely presented group. A finite group presentation of G can be converted to a finite monoid presentation $(\Sigma;R)$ such that R is special and $G \cong M(\Sigma;R)$. In any group, the conjugacy $\sim_c$ is equal to the usual group conjugacy. Therefore, the following proposition follows.

Proposition 3.2. *There is a monoid M defined by a finite special rewriting system such that the c-conjugacy problem for M is undecidable.*

We can say more about the conjugacy relation $\sim_c$ for monoids defined by particular types of rewriting systems.

Monoids defined by special rewriting systems

Lemma 3.3. *Let M be a monoid defined by a special rewriting system $(\Sigma;R)$. If M has a zero, then M is trivial.*

Proof. For $w \in \Sigma^*$, we will write $[w]$ to denote the congruence class $[w]_M$. Suppose that M has a zero, say $[z]$. It has been shown in [39, Lemma 4.3] that any idempotent in M has the form $[a][b]$ with $[b][a] = [1]$. Let $a, b \in \Sigma^*$ be such words for the zero $[z]$. Thus $[1] = [1][1] = ([b][a])([b][a]) = [b]([a][b])[a] = [b][z][a] = [z]$, so M is trivial. □

Lemma 3.3 and (2b) of Theorem 2.2 give the following result.

Proposition 3.4. *In every monoid M defined by a special rewriting system, $\sim_c = \sim_o$.*

Zhang [55, Theorem 3.2] has proved that in every monoid M defined by a special rewriting system, the relations $\sim_l$, $\sim_p$, and $\sim_o$ coincide. This theorem combined with Proposition 3.4 gives the following result.

Proposition 3.5. *In every monoid M defined by a special rewriting system, the relations $\sim_c$, $\sim_l$, $\sim_p$, and $\sim_o$ coincide.*

Proposition 3.5 is not true for the monoids defined by monadic rewriting systems, even when such systems are also finite and confluent. This follows from [48, Example 2.2], as well as from the following example.

Example 3.6. Consider any finite, monadic, and confluent rewriting system $(\Sigma;R)$. Select a letter $a \notin \Sigma$ and let $T = R \cup \{(ax,a),(xa,a),(aa,a) : x \in \Sigma\}$. Then T is also a finite, monadic, and confluent rewriting system. Let M be the monoid defined by $(\Sigma \cup \{a\};T)$.

It is easy to see that $[a]_M$ is a zero in M and that $M \neq \{[a]_M\}$. Hence $\sim_o$ is the universal relation in M, while $\sim_c$ is not universal since, as we have already observed, in any semigroup with zero, the only element $\sim_c$-related to 0 is 0. Hence $\sim_o \neq \sim_c$ in M.

Otto [48, Theorem 3.8] has proved that if M is a monoid defined by a finite, special, and confluent rewriting system, then the o-conjugacy problem for M is decidable. Combining this theorem with Proposition 3.5, we obtain the following result.

Proposition 3.7. *Let M be a monoid defined by a finite, special, and confluent rewriting system. Then the c-conjugacy problem for M is decidable.*

One-relation monoids

A monoid M is called a *one-relation* monoid if it admits a finite presentation with one defining relation, which we will write as $(\Sigma; u = v)$ instead of $(\Sigma, \{(u, v)\})$. Many decision problems have been studied in the class of one-relation monoids. For example, it is decidable whether a one-relation monoid has a zero [13, Proposition 14]. Moreover, a one-relation monoid M containing a zero admits a presentation $(\{a\}; a^{k+1} = a^k)$, where k is a positive integer [13, the proof of Propositin 14].

Let M be the monoid defined by a presentation $(\{a\}; a^{k+1} = a^k)$, where k is a positive integer. For $x \in \{a\}^*$, we write x to mean the congruence class $[x]_M$. With this notation, $M = \{1, a, a^2, \ldots, a^k\}$ and a^k is the zero, which we will write as 0. Let $a^i \in M$ with $a^i \neq 0$, that is, $0 \leq i < k$ (where $a^0 = 1$). Then $a^{k-1} = a^{k-1-i}a^i \in Ma^i - \{0\}$ and $a^{k-1}a^j = 0$ for all $a^j \in M$ such that $a^j \neq 1$. Thus $\mathbb{P}(a^i) = \{1\}$ (see Section 2). It follows that $\sim_c = \{(x, x) : x \in M\}$. On the other hand, $\sim_o = M \times M$ since M has a zero.

By the foregoing argument, if M is a one-relation monoid with zero, then the c-conjugacy and o-conjugacy problems for M are decidable. If M has no zero, then $\sim_c = \sim_o$ by Theorem 2.2. Therefore, the following proposition is true.

Proposition 3.8. *Let M be a one-relation monoid. Then c-conjugacy problem for M is decidable if and only if the o-conjugacy problem for M is decidable.*

Some specific results concerning the decidability of the o-conjugacy problem can be found in the literature. Zhang [55, Theorem 4.5] has shown that the o-conjugacy problem is decidable for one-relation monoids $M(\Sigma; u^n = 1)$, where $n > 1$. The same author [56, Theorem 5.3] has shown that the o-conjugacy problem is decidable for one-relation monoids $M(\Sigma; (uv)^m u = (uv)^n u)$, where $m + n \geq 2$. Thus, by Proposition 3.8, the c-conjugacy problem is also decidable for these types of one-relation monoids.

Monoids defined by finite complete rewriting systems

Recall that a rewriting system (Σ, R) is complete if it is both Noetherian and confluent. Narendran and Otto [42, Lemma 3.6] have constructed a finite complete rewriting system $(\Sigma; R)$ such that the o-conjugacy problem is undecidable for the monoid $M = M(\Sigma; R)$. Using the same monoid and arguing that M cannot have a zero and so $\sim_c = \sim_o$, we get the following result.

Proposition 3.9. *There is a monoid M defined by a finite complete rewriting system such that the c-conjugacy problem for M is undecidable.*

Proof. In this proof we consider the monoid $M = M(\Gamma; T)$ defined by Narendran and Otto in [42, page 35], where the rewriting system $(\Gamma; T)$ is complete [42, Lemmas 3.2 and 3.3]. The alphabet Γ contains a particular letter, denoted z, which occurs only in the following rules that define T: $(zh', h'q_a hz)$ and (zx, z), where h, h', q_a are letters and x ranges over a set of letters that does not include z.

Suppose to the contrary that M has a zero, say $[u]_M$. Since $(\Gamma; T)$ is complete, there is a unique $w \in \Gamma^*$ such that w is irreducible (that is, if $w \underset{R}{\longrightarrow} w'$ then $w' = w$) and $[u]_M = [w]_M$ [12, Theorem 1.1.12]. Since $[w]_M$ is the zero, we have $[wz]_M = [w]_M[z]_M = [w]_M$. However, since w is irreducible and $(zh', h'q_a hz)$ and (zx, z) are the only pairs in T that contain z, it is clear that wz is also irreducible. But this implies $w = wz$, which is a contradiction.

Hence M cannot have a zero, and so $\sim_c = \sim_o$ in M. Since, by [42, Lemma 3.6], M has undecidable o-conjugacy problem, it also has undecidable c-conjugacy problem. $\square$

However, in the class of monoids defined by finite, length-reducing, and confluent rewriting systems, the o-conjugacy problem is decidable [41, Corollary 2.7]. It is also known that the p-conjugacy problem is undecidable in this class [42, Corollary 2.4]. It is an open question if the c-conjugacy problem is decidable in this class of monoids.

Independence of the c-conjugacy problem

The notions $\sim_l$, $\sim_p$, $\sim_o$, and $\sim_c$ of conjugacy for a semigroup S reduce to the group conjugacy when S is a group. For groups, the word problem is reducible to the conjugacy problem [48, page 225]. Therefore, if the conjugacy problem for a group G is decidable, then the word problem for G is also decidable. The situation for monoids is different. Osipova [47] has proved that for finitely presented monoids, the word problem, the p-conjugacy problem, and the o-conjugacy problem are pairwise independent. (Decision problems P_1 and P_2 for finitely presented monoids are *independent* if there exist finitely presented monoids M_1 and M_2 such that for M_1, P_1 is decidable and P_2 undecidable; and for M_2, P_2 is decidable and P_1 is undecidable.) It is therefore of interest to find out if the c-conjugation problem is independent of the three problems considered in [47].

Theorem 3.10. *For finitely presented monoids, the word problem and the c-conjugacy problem are independent.*

Proof. First, there are finitely presented groups (and so finitely presented monoids) with decidable word problem but undecidable conjugacy problem [10, 18].

We will construct a finitely presented monoid for which the converse is true. Let $G = M(\Sigma; R)$ be a finitely presented group with undecidable word problem (see [46]), where $(\Sigma; R)$ is a monoid presentation. Let a, b be symbols not in Σ, and let $M = M(A; T)$ be the monoid defined by the rewriting system $(A; T)$, where

$$\begin{aligned} A &= \Sigma \cup \{a, b\}, \\ T &= R \cup \{(xa, ax) : x \in \Sigma\} \cup \{(bx, b) : x \in \Sigma \cup \{a\}\} \cup \{(xb, b) : x \in \Sigma\} \cup \{(aa, a)\}. \end{aligned}$$

Notice that G is a subgroup of M. The word problem for M is undecidable (since otherwise it would be decidable for G). It is easy to see that M has no zero and that each congruence class $[u] = [u]_M$ has a representative of the form b^p, aw, ab^p, or w, where p is a positive integer and $w \in \Sigma^*$.

Observe that whenever a rewriting rule is applied the number of occurrences of b does not change. Thus, for all $u_1, u_2 \in A^*$, if $[u_1] = [u_2]$, then $|u_1|_b = |u_2|_b$. Let $[u], [v] \in M$. Suppose $[u] \sim_c [v]$. Then $[u][t] = [t][v]$ for some $t \in A^*$. Thus $[ut] = [tv]$, and so $|u|_b = |v|_b$ by the foregoing observation.

Conversely, suppose $|u|_b = |v|_b$. If $|u|_b = |v|_b = 0$, then $[u] \sim_c [v]$ since $[u][ab] = [ab] = [ab][v]$ and $[v][ab] = [ab] = [ab][u]$. Suppose $|u|_b = |v|_b = p > 0$. If $[u] = [v]$, then $[u] \sim_c [v]$. Suppose $[u] \neq [v]$. Then $[u] = b^p$ and $[v] = [ab^p]$, or vice versa. We may assume that $[u] = b^p$ and $[v] = [ab^p]$. Then $[u] \sim_c [v]$ since $[u][b] = [b^{p+1}] = [b][v]$ and $[v][a] = [ab^p] = [a][u]$.

We have proved that for all $u, v \in A^*$, $[u] \sim_c [v]$ if and only if $|u|_b = |v|_b$. Hence the c-conjugacy problem for M is decidable. $\square$

Theorem 3.11. *For finitely presented monoids, the p-conjugacy problem and the c-conjugacy problem are independent.*

Proof. Let $M = M(A; T)$ be the monoid from the proof of Theorem 3.10. For $w \in \Sigma^*$, we will write $[w] = [w]_M$ for the element of the monoid M, and $[w]_G$ for the element of the group G.

Let $u, v \in \Sigma^*$. Suppose $[u] \sim_p [v]$, that is, $[u] = [s][t]$ and $[v] = [t][s]$ for some $s, t \in A^*$. The words s and t cannot contain b since in the rewriting system $(A; T)$ a word with b cannot be reduced to a word without b. But then s and t cannot contain a either since a word with a cannot be reduced to a word without a unless b is also present. It follows that $[u]_G = [s]_G[t]_G$ and $[v]_G = [t]_G[s]_G$, and so $[u]_G \sim_p [v]_G$.

We have proved that for all $u, v \in \Sigma^*$, if $[u] \sim_p [v]$, then $[u]_G \sim_p [v]_G$. The converse is clearly true. Since $\sim_p$ in G is the group conjugacy and G has undecidable conjugacy problem, it follows that the p-conjugacy problem for M is undecidable. We have already established in the proof of Theorem 3.10 that the c-conjugacy problem for M is decidable.

We will now present a monoid that has decidable p-conjugacy problem and undecidable c-conjugacy problem. Osipova [47] has shown that there exists a finitely presented monoid M that has decidable p-conjugacy problem and undecidable l-conjugacy problem. Osipova's proof follows the following steps (we use the original notation): (i) she considers a finitely presented monoid $\Pi_1 = M(\mathcal{U}_1; \mathcal{B}_0)$ with undecidable p-conjugacy problem; (ii) she extends the alphabet $\mathcal{U}_1$ to $\mathcal{U}_3 = \mathcal{U}_1 \cup \{c, d, e_1, \ldots, e_m\}$, where $m = |\mathcal{U}_1| + 2|\mathcal{B}_0|$, and builds a new finitely presented monoid $\Pi_3 = M(\mathcal{U}_3; \mathcal{B}_3)$; (iii) she shows [47, Lemma 4] that, for all words $Q, R \in \mathcal{U}_1^*$, we have $Q \sim_p R$ in Π_1 if and only if there exists $X \in \mathcal{U}_3^*$ such that $XcQd = cRdX$ in Π_3; (iv) she concludes [47, Theorem 2] that the l-conjugacy problem for Π_3 is undecidable; (v) she shows [47, Theorem 3] that the p-conjugacy problem for Π_3 is decidable.

Now, notice that $\sim_p$ is symmetric, and hence, by [47, Lemma 4], for all words $Q, R \in \mathcal{U}_1^*$, we have $Q \sim_p R$ in Π_1 if and only if there exist $X, Y \in \mathcal{U}_3^*$ such that $XcQd = cRdX$ and $YcRd = cQdY$ in Π_3. Equivalently, $Q \sim_p R$ in Π_1 if and only if $cQd \sim_o cRd$ in Π_3. Therefore, Π_3 has undecidable o-conjugacy problem.

The set $\mathcal{B}_3$ of Π_3 has rewriting rules of the form (e_icG_i, ce_i), (e_ib_j, b_je_i), and $(e_id, G_i'de_i)$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$, the b_j are the letters of the alphabet $\mathcal{U}_1$, and the G_i and G_i' are fixed words in $\mathcal{U}_1^*$ [47, pages 70 and 71]. From the form of these rules, we can easily deduce that any two words in $\mathcal{U}_3^*$ that are equal in Π_3 have the same number of occurrences of the letter c. Therefore, Π_3 does not have a zero since the zero element, say $[z]$, would satisfy the identity $[z][c] = [z]$, contradicting the above observation. It follows by Theorem 2.2 that $\sim_o = \sim_c$, and hence Π_3 has undecidable c-conjugacy problem. □

We do not know if the c-conjugacy problem and the o-conjugacy problem are independent in the finitely presented monoids. Consider a finitely presented monoid M without zero that has undecidable c-conjugacy problem (such an M exists by Proposition 3.9). Let M^0 be the monoid M with a zero 0 adjoined. Then M^0 is finitely presented and the c-conjugacy problem for M^0 is undecidable (since for all $a, b \in M$, $a \sim_c b$ in M^0 if and only $a \sim_c b$ in M). On the other hand, the o-conjugacy problem for M^0 is decidable since $\sim_o = M^0 \times M^0$.

Now, suppose M is a finitely presented monoid such that has undecidable o-conjugacy problem and decidable c-conjugacy problem. Then the statement "M has a zero" cannot be proved in ZFC (since if we knew that M has a zero, then the algorithm that always says YES would decide if $[u]_M \sim_o [v]_M$ for all $[u]_M, [v]_M \in M$) and it cannot be disproved in ZFC (since if we knew that M has no zero, then the algorithm that works for $\sim_c$ would also work for $\sim_o$). Hence whether the o-conjugacy problem and the c-conjugacy problem are independent hinges on the answer to the following question.

Question. Does there exist a finitely presented monoid M such that the statement "M has a zero" cannot be proved or disproved in ZFC, the o-conjugacy problem for M is undecidable, and the c-conjugacy problem for M is decidable?

4 Restricted Partial Homomorphisms of Digraphs

The remainder of the paper is devoted to the study of the conjugacy $\sim_c$ in several important semigroups of transformations on a set X (finite or infinite). The main tool in our study will

be the characterization of $\sim_c$ in terms of certain partial homomorphisms of directed graphs (see Theorem 4.7 and Corollary 4.8).

A *directed graph* (or a *digraph*) is a pair $\Gamma = (X, \rho)$ where X is non-empty a set (not necessarily finite) and ρ is a binary relation on X. Any element $x \in X$ is called a *vertex* of Γ, and any pair $(x, y) \in \rho$ is called an *arc* of Γ. We will call a vertex x *terminal* if there is no $y \in X$ such that $(x, y) \in \rho$.

Let $\Gamma_1 = (X_1, \rho_1)$ and $\Gamma_2 = (X_2, \rho_2)$ be digraphs. A mapping $\phi : X_1 \to X_2$ is called a *homomorphism* from Γ_1 to Γ_2 if for all $x, y \in X_1$, if $(x, y) \in \rho_1$, then $(x\phi, y\phi) \in \rho_2$ [24]. Generalizing, a partial mapping ϕ from X_1 to X_2 (that is, a mapping ϕ from some subset of X_1 to X_2) is called a *partial homomorphism* from Γ_1 to Γ_2 if for all $x, y \in X_1$, if $(x, y) \in \rho_1$ and $x, y \in \mathrm{dom}(\phi)$, then $(x\phi, y\phi) \in \rho_2$. (For any mapping $f : A \to B$, we will denote the domain of f by $\mathrm{dom}(f)$ and the image of f by $\mathrm{im}(f)$.)

Definition 4.1. Let $\Gamma_1 = (X_1, \rho_1)$ and $\Gamma_2 = (X_2, \rho_2)$ be digraphs. A partial mapping ϕ from X_1 to X_2 is called a *restrictive partial homomorphism* (or an *rp-homomorphism*) from Γ_1 to Γ_2 if it satisfies the following conditions for all $x, y \in X_1$:

(a) If $(x, y) \in \rho_1$, then $x, y \in \mathrm{dom}(\phi)$ and $(x\phi, y\phi) \in \rho_2$;

(b) If x is a terminal vertex in Γ_1 and $x \in \mathrm{dom}(\phi)$, then $x\phi$ is a terminal vertex in Γ_2.

We say that Γ_1 is *rp-homomorphic* to Γ_2 if there is an rp-homomorphism from Γ_1 to Γ_2.

Clearly, every rp-homomorphism from Γ_1 to Γ_2 is a partial homomorphism from Γ_1 to Γ_2. It is also clear that the composition of rp-homomorphisms is an rp-homomorphism, that is, if $\Gamma_1 = (X_1, \rho_1)$, $\Gamma_2 = (X_2, \rho_2)$, and $\Gamma_3 = (X_3, \rho_3)$ are digraphs, ϕ is an rp-homomorphism from Γ_1 to Γ_2, and ψ is an rp-homomorphism from Γ_2 to Γ_3, then $\phi\psi$ is an rp-homomorphism from Γ_1 to Γ_3.

In picturing directed graphs, we will adopt the convention that the arrows will be deleted with the understanding that the arrow goes up along the edge, to the right if the edge is horizontal, and the arrows go counter-clockwise along a cycle. For example, consider the digraphs $\Gamma_1 = (X_1, \rho_1)$, where $X_1 = \{1, 2, 3, 4\}$ and $\rho_1 = \{(2, 3), (3, 4)\}$, and $\Gamma_2 = (X_2, \rho_2)$, where $X_2 = \{a, b, c, d\}$ and $\rho_2 = \{(a, b), (b, d), (c, d)\}$. Then a mapping presented in Figure 4.1 is a partial homomorphism from Γ_1 to Γ_2 (but not a restricted partial homomorphism), and a mapping from Figure 4.2 is an rp-homomorphism from Γ_1 to Γ_2.

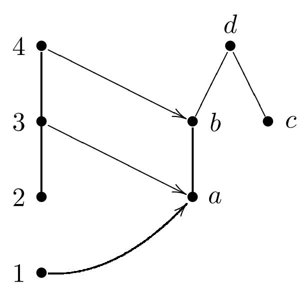

Figure 4.1: A partial homomorphism from Γ_1 to Γ_2.

Let $\alpha \in P(X)$. Then α can be represented by the digraph $\Gamma(\alpha) = (X, \alpha)$, where α is viewed as a binary relation on X. In other words, (x, y) is an arc in $\Gamma(\alpha)$ if and only if $x \in \mathrm{dom}(\alpha)$ and $x\alpha = y$. If $x \in \mathrm{dom}(\alpha)$ and $x\alpha = y$, we will write $x \xrightarrow{\alpha} y$ (or $x \to y$ if no ambiguity arises). For $\alpha \in P(X)$, the set $\mathrm{dom}(\alpha) \cup \mathrm{im}(\alpha)$ will be called the *span* of α and denoted $\mathrm{span}(\alpha)$.

For example, the digraph in Figure 4.3 represents the transformation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \\ 2 & 3 & 1 & 1 & 1 & 5 & 8 & 9 & 10 & \dots \end{pmatrix} \in T(X),$$

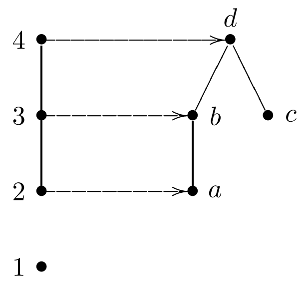

Figure 4.2: An rp-homomorphism from Γ_1 to Γ_2.

where $X = \{1, 2, 3, \ldots\}$ and $T(X)$ is the semigroup of all $\alpha \in P(X)$ such that $\mathrm{dom}(\alpha) = X$.

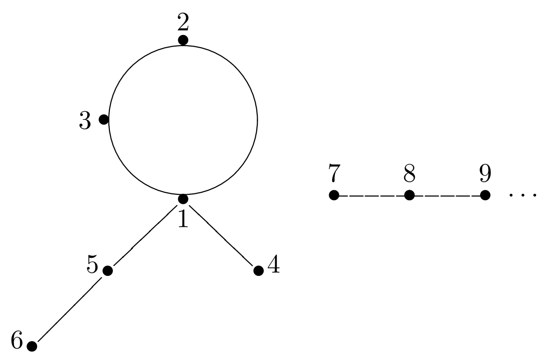

Figure 4.3: The digraph of a transformation.

Definition 4.2. Any $\alpha \in P(X)$ with $\mathrm{im}(\alpha) = \{x\}$ will be called a *constant*. A subsemigroup S of $P(X)$ will be called *constant rich* if for every $x \in X$, there is $\alpha \in S$ such that $\mathrm{im}(\alpha) = \{x\}$.

Among the constant rich subsemigroups of $P(X)$, we have $P(X)$ itself (an all its nonzero ideals), the full transformation semigroup $T(X)$ (and all its ideals), and the symmetric inverse semigroup $\mathcal{I}(X)$ of all injective $\alpha \in P(X)$ (and all its nonzero ideals).

Notation 4.3. From now on, we will fix a nonempty set X and an element $\diamond \notin X$. For $\alpha \in P(X)$ and $x \in X$, we will write $x\alpha = \diamond$ if and only if $x \notin \mathrm{dom}(\alpha)$. We will also assume that $\diamond\alpha = \diamond$. With this notation, it will make sense to write $x\alpha = y\beta$ or $x\alpha \neq y\beta$ ($\alpha, \beta \in P(X)$, $x, y \in X$) even when $x \notin \mathrm{dom}(\alpha)$ or $y \notin \mathrm{dom}(\beta)$.

We will also denote by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ the set of integers, positive integers, and nonnegative integers, respectively, and for semigroups S and T, write $S \leq T$ to mean that S is a subsemigroup of T.

Lemma 4.4. *Let $S \leq P(X)$ such that S is constant rich or $S \leq T(X)$, let $\alpha \in S$ with $\alpha \neq 0$, and $\phi \in S^1$. Then:*

(1) $\phi \in \mathbb{P}^1(\alpha)$ *if and only if* $\mathrm{im}(\alpha) \subseteq \mathrm{dom}(\phi)$.

(2) *If $\phi \in \mathbb{P}^1(\alpha)$ and $\alpha\phi = \phi\beta$ for some $\beta \in S$, then* $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\phi)$ *and for all $x, y \in X$, $x \xrightarrow{\alpha} y$ implies $x\phi \xrightarrow{\beta} y\phi$.*

Proof. If $S \leq T(X)$, then (1) is obvious. Let S be constant rich. Suppose $\phi \in \mathbb{P}^1(\alpha)$. Let $y \in \mathrm{im}(\alpha)$, that is, $y = x\alpha$ for some $x \in \mathrm{dom}(\alpha)$. Since S is constant rich, there is $\gamma \in S$ with

$\mathrm{im}(\gamma) = \{x\}$. Then $\mathrm{im}(\gamma\alpha) = \{y\}$, and so $\gamma\alpha \in S^1\alpha - \{0\}$. Thus $(\gamma\alpha)\phi \neq 0$ (since $\phi \in \mathbb{P}^1(\alpha)$), which is only possible when $y \in \mathrm{dom}(\phi)$. Hence $\mathrm{im}(\alpha) \subseteq \mathrm{dom}(\phi)$.

Conversely, suppose $\mathrm{im}(\alpha) \subseteq \mathrm{dom}(\phi)$. Let $\mu\alpha \in S^1\alpha - \{0\}$. Since $\mu\alpha \neq 0$, there is $x \in X$ such that $x(\mu\alpha) \neq 0$. But then $x(\mu\alpha) = (x\mu)\alpha \in \mathrm{im}(\alpha) \subseteq \mathrm{dom}(\phi)$, and so $x \in \mathrm{dom}((\mu\alpha)\phi)$. Thus $(\mu\alpha)\phi \neq 0$, and so $\phi \in \mathbb{P}^1(\alpha)$. We have proved (1).

To prove (2), suppose $\phi \in \mathbb{P}^1(\alpha)$ and $\alpha\phi = \phi\beta$ for some $\beta \in S$. Let $x, y \in X$ and suppose that $x \xrightarrow{\alpha} y$. Then, since $\alpha\phi = \phi\beta$, we have

$$(x\phi)\beta = x(\phi\beta) = x(\alpha\phi) = (x\alpha)\phi = y\phi. \tag{4.1}$$

By (1), $\mathrm{im}(\alpha) \subseteq \mathrm{dom}(\phi)$, and so $y = x\alpha \in \mathrm{dom}(\phi)$. Then, by (4.1), $x\phi \neq \diamond$, which implies $x \in \mathrm{dom}(\phi)$. It follows that $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\phi)$. Moreover, by (4.1) again, $(x\phi)\beta = y\phi \neq \diamond$, and so $x\phi \xrightarrow{\beta} y\phi$. □

Lemma 4.5. *Let $\alpha, \beta \in P(X)$ and let ϕ be an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Then* $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\phi)$.

Proof. Let $x \in \mathrm{span}(\alpha)$. If $x \in \mathrm{dom}(\alpha)$, then $x \xrightarrow{\alpha} x\alpha$, and so $x, x\alpha \in \mathrm{dom}(\phi)$ by Definition 4.1. If $x \in \mathrm{im}(\alpha)$, then $z \xrightarrow{\alpha} x$ for some $z \in \mathrm{dom}(\alpha)$, and so $z, x \in \mathrm{dom}(\phi)$. Hence $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\phi)$. □

Lemma 4.6. *Let $S \leq P(X)$ such that S is constant rich or $S \leq T(X)$, let $\alpha, \beta \in S$ with $\alpha \neq 0$, and $\phi \in S^1$. Then $\alpha\phi = \phi\beta$ with $\phi \in \mathbb{P}^1(\alpha)$ if and only if ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$.*

Proof. Suppose $\alpha\phi = \phi\beta$ with $\phi \in \mathbb{P}^1(\alpha)$. Let $x, y \in X$ and suppose that $x \xrightarrow{\alpha} y$. Then $x\phi \xrightarrow{\beta} y\phi$ by Lemma 4.4, and so ϕ satisfies (a) of Definition 4.1. Suppose that x is a terminal vertex of $\Gamma(\alpha)$ and $x \in \mathrm{dom}(\phi)$. Then $x\phi \in X$ and $x\alpha = \diamond$. Since $\alpha\phi = \phi\beta$, we have $(x\phi)\beta = (x\alpha)\phi = \diamond\phi = \diamond$, and so $x\phi$ is a terminal vertex in $\Gamma(\beta)$. Hence ϕ satisfies (b) of Definition 4.1. Thus ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$.

Conversely, suppose that ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Let $x \in X$. Suppose $x \notin \mathrm{dom}(\alpha)$. Then $x(\alpha\phi) = (x\alpha)\phi = \diamond\phi = \diamond$. If $x \notin \mathrm{dom}(\phi)$, then $x(\phi\beta) = (x\phi)\beta = \diamond\beta = \diamond$. If $x \in \mathrm{dom}(\phi)$, then, by (b) of Definition 4.1, $x\phi$ is a terminal vertex in $\Gamma(\beta)$, and so $x(\phi\beta) = (x\phi)\beta = \diamond$. Hence, in both cases, $x(\alpha\phi) = x(\phi\beta)$.

Suppose $x \in \mathrm{dom}(\alpha)$ and let $y = x\alpha \in X$. Then $x \xrightarrow{\alpha} y$, and so, by Definition 4.1, $x, y \in \mathrm{dom}(\phi)$ and $x\phi \xrightarrow{\beta} y\phi$. Hence $x(\alpha\phi) = (x\alpha)\phi = y\phi$ and $x(\phi\beta) = (x\phi)\beta = y\phi$, and so $x(\alpha\phi) = x(\phi\beta)$. We have proved that $\alpha\phi = \phi\beta$. Finally, since ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$, we have that $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\phi)$ by Lemma 4.5, and so $\phi \in \mathbb{P}^1(\alpha)$ by Lemma 4.4. □

Theorem 4.7. *Let $S \leq P(X)$ such that S is constant rich or $S \leq T(X)$, let $\alpha, \beta \in S$. Then $\alpha \sim_c \beta$ in S if and only if there are $\phi, \psi \in S^1$ such that ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ and ψ is an rp-homomorphism from $\Gamma(\beta)$ to $\Gamma(\alpha)$.*

Proof. Suppose $\alpha \sim_c \beta$. If $\alpha = 0$, then $\beta = 0$ (since $[0]_c = \{0\}$), and so $\phi = \mathrm{id}_X \in S^1$ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Suppose $\alpha \neq 0$. Since $\alpha \sim_c \beta$, there is $\phi \in \mathbb{P}^1(\alpha)$ such that $\alpha\phi = \phi\beta$, and so ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ by Lemma 4.6. A desired $\psi \in S^1$ exists by symmetry.

Conversely, suppose that desired ϕ and ψ exist. If $x \xrightarrow{\alpha} y$ then $x\phi \xrightarrow{\beta} y\phi$, and if $x \xrightarrow{\beta} y$ then $x\psi \xrightarrow{\alpha} y\psi$. It follows that either $\alpha = \beta = 0$ or $\alpha, \beta \neq 0$. In the former case, we clearly have $\alpha \sim_c \beta$. Suppose $\alpha, \beta \neq 0$. Then, by Lemma 4.6, $\alpha\phi = \phi\beta$ with $\phi \in \mathbb{P}^1(\alpha)$ and $\beta\psi = \psi\alpha$ with $\psi \in \mathbb{P}^1(\beta)$, which implies $\alpha \sim_c \beta$. □

Let $\alpha, \beta \in T(X)$. Then the graph $\Gamma(\alpha)$ has no terminal vertices (if $x \in X$, then $x \xrightarrow{\alpha} x\alpha$), and so every homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ is restrictive. This observation and Theorem 4.7 give us the following corollary.

Corollary 4.8. *Let $S \leq T(X)$, and let $\alpha, \beta \in S$. Then $\alpha \sim_c \beta$ in S if and only if there are $\phi, \psi \in S^1$ such that ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ and ψ is a homomorphism from $\Gamma(\beta)$ to $\Gamma(\alpha)$.*

5 Connected Partial Transformations

In this section, we introduce the concept of connected partial transformation. The definitions and results of this section will be crucial in characterizing conjugacy in various semigroups of transformations.

Definition 5.1. Let $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2 \ldots$ be pairwise distinct elements of X. The following elements of $P(X)$ will be called *basic* partial transformations on X.

(1) A *cycle* of length k ($k \geq 1$), written $(x_0\, x_1 \ldots x_{k-1})$, is an element of $P(X)$ defined by the digraph

$$x_0 \to x_1 \to \cdots \to x_{k-1} \to x_0.$$

(2) A *right ray*, written $[x_0\, x_1\, x_2 \ldots\rangle$, is an element of $P(X)$ defined by the digraph

$$x_0 \to x_1 \to x_2 \to \cdots .$$

(3) A *double ray*, written $\langle \ldots\, x_{-1}\, x_0\, x_1 \ldots\rangle$, is an element of $P(X)$ defined by the digraph

$$\cdots \to x_{-1} \to x_0 \to x_1 \to \cdots .$$

(4) A *left ray*, written $\langle \ldots\, x_2\, x_1\, x_0]$, is an element of $P(X)$ defined by the digraph

$$\cdots \to x_2 \to x_1 \to x_0.$$

(5) A *chain* of length k ($k \geq 1$), written $[x_0\, x_1 \ldots x_k]$, is an element of $P(X)$ defined by the digraph

$$x_0 \to x_1 \to \cdots \to x_k.$$

By a *ray* we will mean a double, right, or left ray.

We note the following:

(i) All basic partial transformations are injective.

(ii) The span of a basic partial transformation is exhibited by the notation. For example, the span of the right ray $[1\,2\,3 \ldots\rangle$ is $\{1, 2, 3, \ldots\}$.

(iii) The left bracket in "$\sigma = [x \ldots$" indicates that $x \notin \mathrm{im}(\sigma)$; while the right bracket in "$\sigma = \ldots x]$" indicates that $x \notin \mathrm{dom}(\sigma)$. For example, for the chain $\sigma = [1\,2\,3\,4]$, $\mathrm{dom}(\sigma) = \{1, 2, 3\}$ and $\mathrm{im}(\sigma) = \{2, 3, 4\}$.

(iv) A cycle $(x_0\, x_1 \ldots x_{k-1})$ differs from the corresponding cycle in the symmetric group of permutations on X in that the former is undefined for every $x \in X - \{x_0, x_1, \ldots, x_{k-1}\}$ while the latter is fixed for every such x.

Definition 5.2. An element $\gamma \in P(X)$ is called *connected* if $\gamma \neq 0$ and for all $x, y \in \mathrm{span}(\gamma)$, $x\gamma^k = y\gamma^m \neq \diamond$ for some integers $k, m \geq 0$ (where $\gamma^0 = \mathrm{id}_X$).

We note that a nonzero $\gamma \in P(X)$ is connected if and only if the underlying undirected graph of the digraph $\Gamma^0(\gamma)$ is connected, where $\Gamma^0(\gamma)$ is the digraph $\Gamma(\gamma)$ with the isolated vertices removed, and that all basic partial transformations are connected.

Definition 5.3. Let $\alpha, \beta \in P(X)$. We say that β is *contained* in α (or α *contains* or *has* β), and write $\beta \sqsubset \alpha$, if $\mathrm{dom}(\beta) \subseteq \mathrm{dom}(\alpha)$ and $x\beta = x\alpha$ for every $x \in \mathrm{dom}(\beta)$. We say that α and β are *disjoint* if $\mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) = \emptyset$; they are *completely disjoint* if $\mathrm{span}(\alpha) \cap \mathrm{span}(\beta) = \emptyset$.

For example, the right ray $[3\,4\,5\,6\ldots\rangle$ and chain $[0\,1\,2\,5]$ in $P(\mathbb{Z})$, where $\mathbb{Z}$ is the set of integers, are disjoint but not completely disjoint. Their join $[3\,4\,5\,6\ldots\rangle \sqcup [0\,1\,2\,5]$ (see Definition 5.4 below) is connected.

Definition 5.4. Let C be a set of pairwise disjoint elements of $P(X)$. The *join* of the elements of C, denoted $\bigsqcup_{\gamma \in C} \gamma$, is an element of $P(X)$ defined by

$$x\left(\bigsqcup_{\gamma\in C}\gamma\right) = \begin{cases} x\gamma & \text{if } x \in \mathrm{dom}(\gamma) \text{ for some } \gamma \in C, \\ \diamond & \text{otherwise.} \end{cases}$$

If $C = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ is finite, we may write $\bigsqcup_{\gamma \in C} \gamma$ as $\gamma_1 \sqcup \gamma_2 \sqcup \cdots \sqcup \gamma_k$.

For a mapping $f : A \to B$ and $A_1 \subseteq A$, we denote by $f|_{A_1}$ the restriction of f to A_1, and by $A_1 f$ the image of A_1 under f.

Proposition 5.5. *Let $\alpha \in P(X)$ with $\alpha \neq 0$. Then there exists a unique set C of pairwise completely disjoint, connected transformations contained in α such that $\alpha = \bigsqcup_{\gamma \in C} \gamma$.*

Proof. Define a relation ρ on $\mathrm{dom}(\alpha)$ by: $(x, y) \in \rho$ if $x\alpha^k = y\alpha^m \neq \diamond$ for some integers $k, m \geq 0$. It is clear that ρ is an equivalence relation on $\mathrm{dom}(\alpha)$. Let J be a complete set of representatives of the equivalence classes of ρ. For every $x \in J$, let $\gamma_x = \alpha|_{x\rho}$, where $x\rho$ is the ρ-equivalence class of x. By the definition of ρ, each such γ_x is connected, and γ_x and γ_y are completely disjoint for all $x, y \in J$ with $x \neq y$. Then the set $C = \{\gamma_x : x \in J\}$ consists of pairwise completely disjoint, connected transformations contained in α, and $\alpha = \bigsqcup_{\gamma \in C} \gamma$.

Suppose D is any set of pairwise completely disjoint, connected transformations contained in α such that $\alpha = \bigsqcup_{\delta \in D} \delta$. Let $\delta \in D$ and let $y \in \mathrm{dom}(\delta)$. Then $y \in x\rho$ for some $x \in J$. We want to prove that $\delta = \gamma_x$. Let $z \in \mathrm{dom}(\delta)$. Since δ is connected, $y\delta^k = z\delta^m \neq \diamond$ for some $k, m \geq 0$. But then, since δ is contained in α, we have $y\alpha^k = z\alpha^m \neq \diamond$. Hence $(y, z) \in \rho$, and so $z \in y\rho = x\rho = \mathrm{dom}(\gamma_x)$. We have proved that $\mathrm{dom}(\delta) \subseteq \mathrm{dom}(\gamma_x)$.

Suppose to the contrary that $\mathrm{dom}(\gamma_x)$ is not included in $\mathrm{dom}(\delta)$, that is, that there is $w \in \mathrm{dom}(\gamma_x)$ such that $w \notin \mathrm{dom}(\delta)$. Since γ_x is connected, $w\gamma_x^p = x\gamma_x^q \neq \diamond$ for some $p, q \geq 0$. Let $y_i = y\gamma_x^i = y\alpha^i$ and $w_j = w\gamma_x^j = w\alpha^j$ for $i = 0, 1, \ldots, p$ and $j = 0, 1, \ldots, q$. Then $y_p = w_q$ and let $u = y_p = w_q$. With this notation, in the digraph $\Gamma(\alpha)$, we have

$$y = y_0 \to y_1 \to \cdots \to y_p = u \text{ and } w = w_0 \to w_1 \to \cdots \to w_q = u.$$

Since $w \in \mathrm{dom}(\gamma_x) \subseteq \mathrm{dom}(\alpha)$, there is $\delta_1 \in D$ such that $w \in \mathrm{dom}(\delta_1)$. We claim that $\{y_0, y_1, \ldots, y_{p-1}\} \subseteq \mathrm{dom}(\delta)$. If not, then, since $y_0 = y \in \mathrm{dom}(\delta)$, there would be $i \in \{0, \ldots, p-2\}$ such that $y_i \in \mathrm{dom}(\delta)$ and $y_{i+1} \notin \mathrm{dom}(\delta)$. But $y_{i+1} \in \mathrm{dom}(\alpha)$, and so $y_{i+1} \in \mathrm{dom}(\delta_2)$ for some $\delta_2 \in D$. We would then have $\delta \neq \delta_2$ and $y_{i+1} \in \mathrm{span}(\delta) \cap \mathrm{span}(\delta_2)$, which is impossible since δ and δ_2 are completely disjoint. The claim has been proved. By the same argument applied to δ_1 and $\{w_0, w_1, \ldots, w_{q-1}\}$, we obtain $\{w_0, w_1, \ldots, w_{q-1}\} \subseteq \mathrm{dom}(\delta_1)$. Thus

$$y_{p-1}\delta = y_{p-1}\alpha = y_p = u = w_q = w_{q-1}\alpha = w_{q-1}\delta_1.$$

Thus we have $\delta \neq \delta_1$ with $u \in \mathrm{im}(\delta) \cap \mathrm{im}(\delta_1)$, which is a contradiction since δ and δ_1 are completely disjoint. We have proved that $\mathrm{dom}(\gamma_x) \subseteq \mathrm{dom}(\delta)$, and so $\mathrm{dom}(\delta) = \mathrm{dom}(\gamma_x)$. Now for all $v \in \mathrm{dom}(\delta) = \mathrm{dom}(\gamma_x)$, we have $v\delta = v\alpha = v\gamma_x$, and so $\delta = \gamma_x \in C$. We have proved that $D \subseteq C$.

For the reverse inclusion, let γ_x be an arbitrary element of C. Select $y \in \mathrm{dom}(\gamma_x)$. Then, there is $\delta \in D$ such that $y \in \mathrm{dom}(\delta)$. By the foregoing argument, we have $\delta = \gamma_x$, and so $\gamma_x \in D$. Hence $C \subseteq D$, and so $D = C$. We have proved that the set C is unique, which completes the proof. $\square$

Any element of the set C from Proposition 5.5 will be called a *connected component* of α. We note that the connected components of α correspond to the connected components of the underlying undirected graph of $\Gamma(\alpha)$ that are not isolated vertices.

Definition 5.6. Let $\alpha \in P(X)$ and let μ be a basic partial transformation contained in α. We say that μ is *maximal* in α if for every $x \in \mathrm{span}(\mu)$, $x \notin \mathrm{dom}(\mu)$ implies $x \notin \mathrm{dom}(\alpha)$, and $x \notin \mathrm{im}(\mu)$ implies $x \notin \mathrm{im}(\alpha)$. Note that if μ is a cycle or a double ray, then μ is always maximal in α.

For example, consider $\alpha = [3\,4\,5\,6\ldots\rangle \sqcup [0\,1\,2\,5] \in P(\mathbb{Z})$. Then α contains infinitely many right rays, for example $[2\,5\,6\,7\ldots\rangle$, but only two of them, namely $[3\,4\,5\,6\ldots\rangle$ and $[0\,1\,2\,5\,6\,7\ldots\rangle$ are maximal. Also, α contains infinitely many chains, for example $[3\,4\,5\,6]$, but none of them is maximal.

We will now establish which combinations of basic partial transformations can occur in a connected element of $P(X)$.

Lemma 5.7. *Let $\gamma \in P(X)$ be connected. Then:*

(1) *If γ has a cycle $(x_0\,x_1\,\ldots\,x_{k-1})$, then for every $x \in \mathrm{dom}(\gamma)$, $x\gamma^m = x_0$ for some $m \geq 0$.*

(2) *If γ has a right ray $[x_0\,x_1\,x_2\ldots\rangle$ or a double ray $\langle\ldots\,x_{-1}\,x_0\,x_1\ldots\rangle$, then for every $x \in \mathrm{dom}(\gamma)$, $x\gamma^m = x_i$ for some $m, i \geq 0$.*

(3) *If γ has a maximal chain $[x_k\,\ldots\,x_1\,x_0]$ or a maximal left ray $\langle\ldots\,x_2\,x_1\,x_0]$, then for every $x \in \mathrm{span}(\gamma)$, $x\gamma^m = x_0$ for some $m \geq 0$.*

Proof. Suppose γ has a cycle $(x_0\,x_1\,\ldots\,x_{k-1})$ and let $x \in \mathrm{dom}(\gamma)$. Since γ is connected, $x\gamma^p = x_0\gamma^q$ for some $p, q \geq 0$. Since x_0 lies on the cycle $(x_0\,x_1\,\ldots\,x_{k-1})$, we may assume that $0 \leq q \leq k-1$. Thus for $m = p + k - q$, we have

$$x\gamma^m = x\gamma^{p+k-q} = (x\gamma^p)\gamma^{k-q} = (x_0\gamma^q)\gamma^{k-q} = x^q\gamma^{k-q} = x_0.$$

Suppose γ has a right ray $[x_0\,x_1\,x_2\ldots\rangle$ and let $x \in \mathrm{dom}(\gamma)$. Since γ is connected, $x\gamma^m = x_0\gamma^i = x_i$ for some $m, i \geq 0$. A proof in the case of a double ray is the same.

Suppose γ has a chain $[x_k\,\ldots\,x_1\,x_0]$ and let $x \in \mathrm{span}(\gamma)$. Since γ is connected, $x\gamma^p = x_0\gamma^q \neq \diamond$ for some $p, q \geq 0$. Note that q must be 0 (since $x_0\gamma^q = \diamond$ for every $q \geq 1$). Thus $x\gamma^p = x_0\gamma^0 = x_0$. The proof in the case of a maximal left ray is the same. $\square$

Proposition 5.8. *Let $\gamma \in P(X)$ be connected. Then:*

(1) *If γ has a cycle, then the cycle is unique and γ does not have any double rays or rights rays or maximal chains or maximal left rays.*

(2) *If γ has a double ray, then it does not have any maximal chains or maximal left rays.*

(3) *If γ has a right ray but no double rays, then it has a maximal right ray and it does not have any left rays or maximal chains.*

(4) *If γ has a chain but no cycles or rays, then it has a maximal chain.*

(5) *If γ has a left ray but no cycles or double rays, then it has a maximal left ray.*

Proof. Suppose that γ has a cycle, say $\theta = (x_0\,x_1\,\ldots\,x_{k-1})$. Let $\vartheta = (y_0\,y_1\,\ldots\,y_{m-1})$ be any cycle in γ. We want to prove that $\theta = \vartheta$. We may assume that $k \leq m$. By Lemma 5.7, $y_0\gamma^p = x_0$ for some $p \geq 0$. On the other hand, $y_0\gamma^m = y_j$ for some $j \in \{0, \ldots, m-1\}$, and so $x_0 = y_j$. Since we can rewrite ϑ as $(y_j\,y_{j+1}\,\ldots\,y_{j-1})$, we may assume that $y_j = y_0$, so $x_0 = y_0$. But then $x_i = x_0\gamma^i = y_0\gamma^i = y_i$ for every $i \in \{0, \ldots, k-1\}$ and $y_{k-1}\gamma = x_{k-1}\gamma = x_0 = y_0$. It follows that $k = m$ and $\theta = \vartheta$. We have proved that a cycle in γ is unique.

Suppose that γ with a cycle $(x_0\, x_1 \ldots x_{k-1})$ also has a double ray, say $\langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle$. By Lemma 5.7, $y_0\gamma^m = x_0$ for some $m \geq 0$. But then $y_0\gamma^{m+k} = (y_0\gamma^m)\gamma^k = x_0\gamma^k = x_0 = y_0$, which is a contradiction since $y_0\gamma^{m+k} = y_{m+k} \neq y_0$ (since $m \geq 0$ and $k \geq 1$). Thus γ does not have a double ray. Proofs that γ with a cycle cannot have a right ray or a maximal chain or a maximal left ray are similar. This proves (1).

To prove (2), suppose that γ has a double ray, say $\langle \ldots\, x_{-1}\, x_0\, x_1 \ldots\rangle$. Suppose γ has a maximal left ray, say $\langle \ldots\, y_2\, y_1\, y_0]$. By Lemma 5.7, $x_0\gamma^m = y_0$ for some $m \geq 0$. But then $x_0\gamma^{m+1} = \diamond$, which is a contradiction since $x_0\gamma^{m+1} = x_{m+1} \neq \diamond$. A proof that γ with a double ray cannot have a maximal chain is similar.

To prove (3), let $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ be a right ray in α. If η is not maximal, then $x_{-1}\gamma = x_0$ for some $x_{-1} \in X - \{x_0, x_1, \ldots\}$. (If $x_{-1} = x_i$ for some $i \geq 0$, then γ would have a cycle, which is impossible by (1).) Thus $\eta_1 = [x_{-1}\, x_0\, x_1\, x_2 \ldots\rangle$ is a right ray in α. If η_1 is not maximal, then $x_{-2}\gamma = x_{-1}$ for some $x_{-2} \in X - \{x_{-1}, x_0, x_1, \ldots\}$, and so $\eta_2 = [x_{-2}\, x_{-1}\, x_0\, x_1\, x_2 \ldots\rangle$ is a right ray in α. Continuing this way, we must arrive at a maximal right ray in α (after finitely many steps) since otherwise α would have a double ray.

Suppose γ has a left ray, say $\langle \ldots\, y_2\, y_1\, y_0]$. Since γ is connected, $x_0\gamma^k = y_0\gamma^m$ for some $k, m \geq 0$. But then $\langle \ldots\, y_2\, y_1\, y_0\, y_0\gamma \ldots y_0\gamma^m\, x_{k+1}\, x_{k+2} \ldots\rangle$ is a double ray in α, which is a contradiction.

Suppose γ has a maximal chain, say $[y_0\, y_1 \ldots y_k]$. By Lemma 5.7, $x_0\gamma^m = y_k$ for some $m \geq 0$. But then $x_0\gamma^{m+1} = \diamond$, which is a contradiction since $x_0\gamma^{m+1} = x_{m+1} \neq \diamond$. We have proved (3).

To prove (4), let $\lambda = [x_0\, x_1 \ldots x_k]$ be a chain in α. If $x_0 \in \mathrm{im}(\alpha)$, then, since α has no left rays, we can use the argument as in the proof of (3) for a right ray to extend λ to a chain $\lambda' = [x_{-m} \ldots x_{-1}\, x_0\, x_1 \ldots x_k]$ such that $x_{-m} \notin \mathrm{im}(\alpha)$. Similarly, since α has no right rays or cycles, we can extend λ' to a chain $\lambda'' = [x_{-m}\, x_{-m+1} \ldots x_{-1}\, x_0\, x_1 \ldots x_k\, x_{k+1} \ldots x_{k+p}]$ such that $x_{k+p} \notin \mathrm{dom}(\alpha)$. Then λ'' is a maximal chain in α. We have proved (4). The proof of (5) is similar. $\square$

Remark 5.9. It follows from Proposition 5.8 that as far as the types of basic transformations go, a connected $\gamma \in P(X)$ can contain:

(1) A single cycle and no double rays or right rays or maximal chains or maximal left rays (see Figure 5.1);

(2) A double ray but no cycles or maximal chains or maximal left rays (see Figure 5.2);

(3) A maximal right ray but no cycles or double rays or left rays or maximal chains (see Figure 5.3);

(4) A maximal left ray but no cycles or double rays or right rays (see Figure 5.4 and Definition 5.10);

(5) A maximal chain but no cycles or rays (see Figure 5.5 and Definition 5.10).

We note that the uniqueness applies only to a cycle. A connected γ can have any number (finite or infinite) of (maximal) chains or (maximal) rays of any type.

For our purposes, it will not be necessary to distinguish connected partial transformations that have double rays only or left rays only. (In other words, if a connected $\gamma \in P(X)$ has a double ray, then it will not matter whether it has a maximal right ray as well; similarly, if it has a maximal left ray, then it will not matter whether it has a maximal chain as well.) However, we will need to distinguish connected transformations that have right rays only, and connected transformations that have chains only.

Definition 5.10. Let $\gamma \in P(X)$ be connected. If γ satisfies (3) of Remark 5.9, we will say that γ is of (or has) *type rro* ("right rays only"). If γ satisfies (5) of Remark 5.9, we will say that γ is of type *cho* ("chains only").

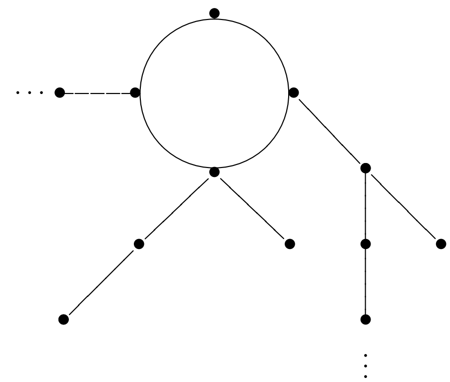

Figure 5.1: A connected partial transformation with a cycle.

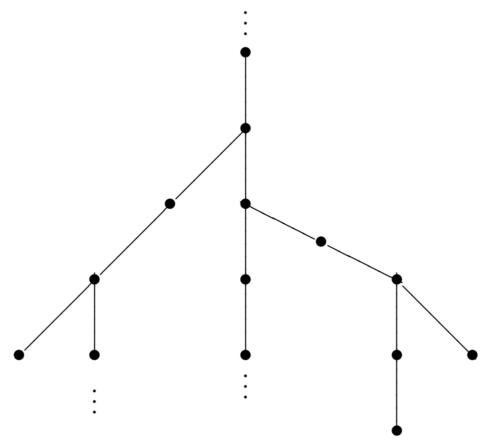

Figure 5.2: A connected partial transformation with a double ray.

Lemma 5.11. *Let $\gamma \in P(X)$ be connected such that γ contains a maximal left ray or it is of type cho. Then there is a unique $x_0 \in \mathrm{span}(\gamma)$ such that $x_0 \in \mathrm{im}(\gamma) - \mathrm{dom}(\gamma)$.*

Proof. Suppose γ contains a maximal left ray, say $\lambda = \langle \dots x_2\, x_1\, x_0]$. Then $x_0 \in \mathrm{im}(\gamma) - \mathrm{dom}(\gamma)$. Let y be any element in $\mathrm{im}(\gamma) - \mathrm{dom}(\gamma)$. Then $z\gamma = y$ for some $z \in \mathrm{dom}(\gamma)$. By Lemma 5.7, $z\gamma^m = x_0$ for some $m \geq 0$. Since $z\gamma = y \notin \mathrm{dom}(\gamma)$ and $z \neq x_0$, we must have $m = 1$. But then $y = z\gamma = x_0$. This concludes the proof in the case when γ has a maximal left ray. A proof in the case when γ is of type *cho* is similar. ☐

For integers a and b, we write $a \,|\, b$ if a divides b, that is, if $b = ak$ for some integer k. For integers a and n with $n \geq 1$, we denote by $\mathrm{mod}(a, n)$ the unique integer r in $\{0, 1, \dots, n-1\}$ such that $a \equiv r \pmod{n}$. We note that

$$\mathrm{mod}(a+1, n) = \begin{cases} \mathrm{mod}(a, n) + 1 & \text{if } \mathrm{mod}(a, n) \neq n-1, \\ 0 & \text{if } \mathrm{mod}(a, n) = n-1. \end{cases} \tag{5.1}$$

Proposition 5.12. *Let $\gamma, \delta \in P(X)$ be connected such that γ has a cycle $(x_0\, x_1 \dots x_{k-1})$. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ if and only if δ has a cycle $(y_0\, y_1 \dots y_{m-1})$ such that $m \,|\, k$.*

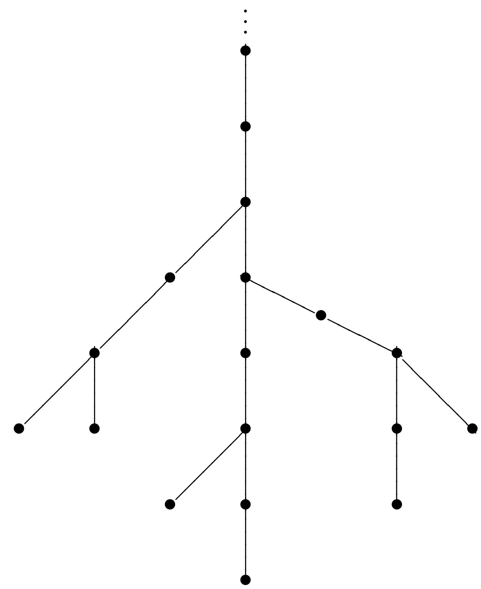

Figure 5.3: A connected partial transformation of type *rro*.

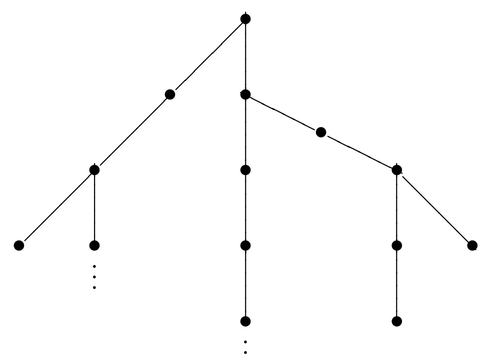

Figure 5.4: A connected partial transformation with a maximal left ray.

Proof. Suppose there is an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$. Let $y_i = x_i\phi$ for $i = 0, 1, \ldots, k-1$. Then $y_0 \xrightarrow{\delta} y_1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} y_{k-1} \xrightarrow{\delta} y_0$, and so $y_0\delta^k = y_0$. Let m be the smallest integer in $\{1, 2, \ldots, k\}$ such that $y_0\delta^m = y_0$. Then $(y_0\, y_1 \ldots\, y_{m-1})$ is a cycle in δ. By the Division Algorithm, $k = mq + r$ for some $q, r \in \mathbb{N}$ with $0 \le r < m$. Since $y_0\delta^m = y_0$, we have $y_0\delta^{mq} = y_0$, and so $y_0 = y_0\delta^k = (y_0\delta^{mq})\delta^r = y_0\delta^r$. Thus $r = 0$ by the definition of m, and so $k = mq$, that is, $m \,|\, k$.

Conversely suppose that δ has a desired cycle. We will define an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$ such that $\mathrm{dom}(\phi) = \mathrm{dom}(\gamma)$ and $\mathrm{im}(\phi) = \{y_0, y_1, \ldots, y_{m-1}\}$. (Note that $\mathrm{dom}(\gamma) = \mathrm{span}(\gamma)$ since γ has a cycle.) For $x \in \mathrm{dom}(\gamma)$, let p_x be the smallest nonnegative integer such that $x\gamma^{p_x} = x_0$ (such p_x exists by Lemma 5.7), and let $q_x = \mathrm{mod}(-p_x, m)$. Define ϕ on $\mathrm{dom}(\gamma)$ by $x\phi = y_{q_x}$. Suppose $x \xrightarrow{\gamma} z$. We consider two possible cases.

Case 1. $x = x_0$.

Then $p_x = 0$, $z = x\gamma = x_0\gamma = x_1$, and $p_z = k - 1$. Thus $q_x = \mathrm{mod}(0, m) = 0$ and

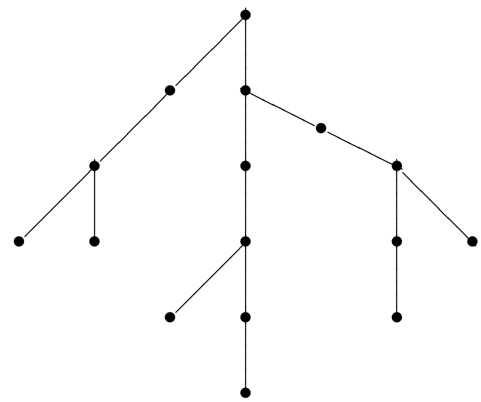

Figure 5.5: A connected partial transformation of type *cho*.

$q_z = \mathrm{mod}(-k+1, m) = 1$ (since $m \mid k$, and so $-k \equiv 0 \pmod{m}$). Hence $x\phi = y_0 \xrightarrow{\delta} y_1 = z\phi$.

Case 2. $x \neq x_0$.

Then, since $x \xrightarrow{\gamma} z$, we have $p_z = p_x - 1$, and so

$$q_z = \mathrm{mod}(-p_z, m) = \mathrm{mod}(-p_x + 1, m). \tag{5.2}$$

Suppose $q_x = \mathrm{mod}(-p_x, m) \neq m-1$. Then, by (5.1) and (5.2), $q_z = \mathrm{mod}(-p_x+1, m) = \mathrm{mod}(-p_x, m) + 1 = q_x + 1$, and so

$$x\phi = y_{q_x} \xrightarrow{\delta} y_{q_x+1} = y_{q_z} = z\phi.$$

Suppose $q_x = \mathrm{mod}(-p_x, m) = m-1$. Then $-p_x \equiv -1 \pmod{m}$, and so $p_x \equiv 1 \pmod{m}$. Thus $p_x = tm+1$ for some integer t, and so $p_z = p_x - 1 = tm$. Hence $q_z = \mathrm{mod}(-p_z, m) = \mathrm{mod}(-tm, m) = 0$, and so

$$x\phi = y_{q_x} = y_{m-1} \xrightarrow{\delta} y_0 = y_{q_z} = z\phi.$$

Thus, in both cases, $x\phi \xrightarrow{\delta} x\phi$, and so ϕ is an rp-homomorphism. (Condition (b) of Definition 4.1 is satisfied since $\Gamma(\gamma)$ does not have any terminal vertices.) $\square$

Lemma 5.13. *Let $\gamma, \delta \in P(X)$ be connected such that δ has a cycle $(y_0\, y_1 \ldots y_{m-1})$. Suppose γ has a double or γ is of type rro. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$.*

Proof. Suppose γ has a double ray $\mu = \langle \ldots x_{-1}\, x_0\, x_1 \ldots \rangle$. We will define an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$ such that $\mathrm{dom}(\phi) = \mathrm{dom}(\gamma)$ and $\mathrm{im}(\phi) = \{y_0, y_1, \ldots, y_{m-1}\}$. For $x \in \mathrm{dom}(\gamma)$, let p_x be the smallest nonnegative integer such that $x\gamma^{p_x} = x_i$ for some i (such p_x exists by Lemma 5.7), and let $q_x = \mathrm{mod}(i - p_x, m)$. Define ϕ on $\mathrm{dom}(\gamma)$ by $x\phi = y_{q_x}$. Suppose $x \xrightarrow{\gamma} z$. We consider two possible cases.

Case 1. $x = x_i$ for some $i \in \mathbb{Z}$.

Then $p_x = 0$, $z = x\gamma = x_i\gamma = x_{i+1}$, and $p_z = 0$. Thus $q_x = \mathrm{mod}(i, m)$ and $q_z = \mathrm{mod}(i+1, m)$. If $q_x \neq m-1$, then $q_z = q_x + 1$, and so $x\phi = y_{q_x} \xrightarrow{\delta} y_{q_x+1} = y_{q_z} = z\phi$. if $q_x = m-1$, then $q_z = 0$, and so $x\phi = y_{q_x} = y_{m-1} \xrightarrow{\delta} y_0 = y_{q_z} = z\phi$.

Case 2. $x \neq x_i$ for every $i \in \mathbb{Z}$.

Then, since $x \xrightarrow{\gamma} z$, we have $p_z = p_x - 1$ with $x\gamma^{p_x} = z\gamma^{p_z} = i$, and so

$$q_z = \mathrm{mod}(i - p_z, m) = \mathrm{mod}(i - p_x + 1, m). \tag{5.3}$$

If $q_x \neq m-1$, then, by (5.1) and (5.3), $q_z = \mathrm{mod}(i - p_x + 1, m) = \mathrm{mod}(i - p_x, m) + 1 = q_x + 1$, and so $x\phi = y_{q_x} \xrightarrow{\delta} y_{q_x+1} = y_{q_z} = z\phi$. If $q_x = m-1$, then $q_z = 0$, and so again $x\phi \xrightarrow{\delta} z\phi$.

Hence, since $\Gamma(\gamma)$ has no terminal vertices, ϕ is an rp-homomorphism. The proof in the case when γ has type *cho* is similar. $\square$

Lemma 5.14. *Let $\gamma, \delta \in P(X)$ be connected. Suppose that δ has a double ray and γ either has a double ray or has type rro. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$.*

Proof. Suppose γ has a double chain, say $\langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$, and let $\langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ be a double ray in δ. We will define an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$ such that $\mathrm{dom}(\phi) = \mathrm{dom}(\gamma)$ and $\mathrm{im}(\phi) = \{\dots, y_{-1}, y_0, y_1, \dots\}$. (Note that $\mathrm{dom}(\gamma) = \mathrm{span}(\gamma)$ since γ has a double ray or it is of type *rro*.) For $x \in \mathrm{dom}(\gamma)$, let p_x be the smallest nonnegative integer such that $x\gamma^{p_x} = x_i$ for some integer i. Define ϕ on $\mathrm{dom}(\gamma)$ by $x\phi = y_{i-p_x}$ where $x\gamma^{p_x} = x_i$. Suppose $x \xrightarrow{\gamma} z$. We consider two possible cases.

Case 1. $x = x_i$ for some integer i.

Then $p_x = 0$, $z = x\gamma = x_i\gamma = x_{i+1}$, and $p_z = 0$. Thus

$$x\phi = y_{i-p_x} = y_i \xrightarrow{\delta} y_{i+1} = y_{i+1-p_z} = z\phi.$$

Case 2. $x \neq x_i$ for every integer i.

Then, since $x \xrightarrow{\gamma} z$, we have $p_z = p_x - 1$ and $x\gamma^{p_x} = z\gamma^{p_z} = x_i$ for some i. Thus

$$x\phi = y_{i-p_x} \xrightarrow{\delta} y_{i-p_x+1} = y_{i-p_z} = z\phi.$$

Thus, in both cases, $x\phi \xrightarrow{\delta} z\phi$, and so ϕ is an rp-homomorphism since $\Gamma(\gamma)$ does not have any terminal vertices. The proof in the case when γ has type *rro* is similar. $\square$

For a mapping $f : A \to B$ and $b \in B$, we denote by bf^{-1} the preimage of b under f, that is, $bf^{-1} = \{a \in A : af = b\}$.

Definition 5.15. Let $\gamma \in P(X)$ be connected of type *rro* (right rays only) or *cho* (chains only). For every ordinal μ, we will define a set (possibly empty) $A_\mu \subseteq \mathrm{span}(\gamma)$. We proceed by transfinite recursion. We define $A_0 = \{x \in \mathrm{span}(\gamma) : x \notin \mathrm{im}(\gamma)\}$. Let μ be an ordinal with $\mu > 0$ and suppose that A_λ has been defined for all ordinals $\lambda < \mu$.

Suppose $\mu = \nu + 1$ for some ordinal ν. We define A_μ as the set of all $x \in \mathrm{span}(\gamma)$ such that:

(a) $x \notin A_\lambda$ for every $\lambda < \mu$;

(b) For all $y \in x\gamma^{-1}$, $y \in A_\lambda$ for some $\lambda < \mu$; and

(c) For some $y \in x\gamma^{-1}$, $y \in A_\nu$.

Suppose μ is a limit ordinal. We define A_μ as the set of all $x \in \mathrm{span}(\gamma)$ such that:

(a) $x \notin A_\lambda$ for every $\lambda < \mu$;

(b) For all $y \in x\gamma^{-1}$, $y \in A_\lambda$ for some $\lambda < \mu$; and

(c) $\mu = \sup\{\lambda < \mu : y \in A_\lambda \text{ for some } y \in x\gamma^{-1}\}$.

Lemma 5.16. *Let $\gamma \in P(X)$ be connected of type rro or cho. Then for every $x \in \mathrm{span}(\gamma)$, there is a unique ordinal μ such that $x \in A_\mu$.*

Proof. Let $x \in \operatorname{span}(\gamma)$, and let μ and λ be ordinals such that $\lambda < \mu$. If $x \in A_\mu$, then $x \notin A_\lambda$ by the definition of A_μ. This proves uniqueness.

Suppose to the contrary that $x \notin A_\mu$ for every ordinal μ. Then $x \in \operatorname{im}(\gamma)$ (since otherwise x would be in A_0), and so the preimage $x\gamma^{-1}$ is not empty. Since $x \notin A_\mu$ for every ordinal μ, there must be some $y_1 \in x\gamma^{-1}$ such that $y_1 \notin A_\lambda$ for every ordinal λ. (Indeed, if such y_1 did not exist, then x would be an element of A_μ for $\mu = \nu$ or $\mu = \nu + 1$ where $\nu = \sup\{\lambda : y \in A_\lambda \text{ for some } y \in x\gamma^{-1}\}$.) Continuing this way, we obtain a sequence $y_1, y_2, y_3, \ldots$ such that $\cdots \xrightarrow{\gamma} y_3 \xrightarrow{\gamma} y_2 \xrightarrow{\gamma} y_1 \xrightarrow{\gamma} x$. But this is a contradiction since then γ would contain a cycle or a left ray. Hence $x \in A_\mu$ for some μ. □

Definition 5.17. Let $\gamma \in P(X)$ be connected of type *rro* or *cho*, and let $x \in \operatorname{span}(\gamma)$. The unique ordinal μ such that $x \in A_\mu$ will be called the *order* of x and denoted $o_\gamma(x)$ (or $o(x)$ if γ is clear from the context).

We note that for every $x \in \operatorname{span}(\gamma)$ with $o(x) > 0$, $o(y) < o(x)$ for every $y \in x\gamma^{-1}$, and if $\nu = \sup\{o(y) : y \in x\gamma^{-1}\}$ then

$$o(x) = \begin{cases} \nu + 1 & \text{if } o(y) = \nu \text{ for some } y \in x\gamma^{-1}, \\ \nu & \text{if } o(y) < \nu \text{ for every } y \in x\gamma^{-1}. \end{cases} \tag{5.4}$$

Example 5.18. Let $X = \{x_0, x_1, x_2, \ldots, y_0, y_1, y_2, \ldots\}$ and let

$$\gamma = [x_0\, x_1\, x_2\, x_3 \ldots\rangle \sqcup [y_0\, x_2] \sqcup [y_1\, y_2\, x_2] \sqcup [y_3\, y_4\, y_5\, x_2] \sqcup [y_6\, y_7\, y_8\, y_9\, x_2] \sqcup \cdots \in P(X).$$

Then γ is connected of type *rro* and we have: $o(x_0) = 0$, $o(x_1) = 1$, and $o(x_{2+i}) = \omega + i$ for every $i \geq 0$, where ω is the smallest infinite ordinal. We also have: $o(y_6) = 0$, $o(y_7) = 1$, $o(y_8) = 2$, and $o(y_9) = 3$.

Example 5.19. Let $X = \{y_0, y_1, y_2, \ldots\} \cup \bigcup_{i=0}^{\infty} Z_i$, where $Z_i = \{z_0^i, z_1^i, z_2^i, \ldots\}$. For every integer $i \geq 0$, let

$$\delta_i = [z_1^i\, z_0^i] \sqcup [z_2^i\, z_3^i\, z_0^i] \sqcup [z_4^i\, z_5^i\, z_6^i\, z_0^i] \sqcup [z_7^i\, z_8^i\, z_9^i\, z_{10}^i\, z_0^i] \sqcup \cdots \in P(X).$$

Then each δ_i is connected of type *cho* and $o_\delta(z_0^i) = \omega$. Further, let

$$\gamma = (\delta_0 \sqcup [z_0^0\, y_0]) \sqcup (\delta_1 \sqcup [z_0^1\, y_1\, y_0]) \sqcup (\delta_2 \sqcup [z_0^2\, y_2\, y_3\, y_0]) \sqcup (\delta_3 \sqcup [z_0^3\, y_4\, y_5\, y_6\, y_0]) \sqcup \cdots \in P(X).$$

Then γ is connected of type *cho* and $o_\gamma(y_0) = \omega + \omega = 2\omega$.

Definition 5.20. Let $\gamma \in P(X)$ be connected such that γ has a maximal left ray or is of type *cho*. Then, by Lemma 5.11, there is a unique x_0 in $\operatorname{im}(\gamma) - \operatorname{dom}(\gamma)$. We will call this unique element the *root* of γ.

Notation 5.21. Let $\gamma \in P(X)$ be connected and let $x \in \operatorname{span}(\gamma)$. We denote by $\downarrow\! x$ the set of all $y \in \operatorname{span}(\gamma)$ such that $x = y\gamma^m$ for some $m \geq 0$. If $x \in \operatorname{im}(\gamma)$, we denote by γ_x the restriction of γ to $\downarrow\! x - \{x\}$. Note that γ_x is connected and it either contains a maximal left ray or is of type *cho*, and that, in either case, x is the root of γ_x.

Lemma 5.22. *Let $\gamma, \delta \in P(X)$ be connected such that γ is of type rro or cho and δ is contained in γ. Then for every $x \in \operatorname{span}(\delta)$:*

(1) $o_\delta(x) \leq o_\gamma(x)$.

(2) *If $\delta = \gamma_z$ for some $z \in \operatorname{im}(\gamma)$, then $o_\delta(x) = o_\gamma(x)$.*

Proof. First note that δ must be of type *rro* or *cho*. Let $x \in \operatorname{span}(\delta)$. To prove (1), we proceed by transfinite induction on $o_\delta(x)$. The result is clearly true if $o_\delta(x) = 0$. Let $o_\delta(x) = \mu > 0$ and suppose that for all $y \in \operatorname{span}(\delta)$, if $o_\delta(y) < \mu$, then $o_\delta(y) \leq o_\gamma(y)$. Let $\nu = \sup\{o_\delta(y) : y \in x\delta^{-1}\}$, $\mu_1 = o_\gamma(x)$, and $\nu_1 = \sup\{o_\gamma(y) : y \in x\gamma^{-1}\}$. Since $x\delta^{-1} \subseteq x\gamma^{-1}$ (because δ is contained in

γ) and $o_\delta(y) \le o_\gamma(y)$ for every $y \in x\delta^{-1}$ (by the inductive hypothesis since $o_\delta(y) < o_\delta(x)$ for every such y), we have $\nu \le \nu_1$. Suppose $o_\delta(y) = \nu$ for some $y \in x\delta^{-1}$. Then $\mu = \nu + 1$ and $\nu = o_\delta(y) \le o_\gamma(y) < \mu_1$, and so $\mu = \nu + 1 \le \mu_1$. Suppose $o_\delta(y) < \nu$ for every $y \in x\delta^{-1}$. Then $\mu = \nu \le \nu_1 \le \mu_1$. Hence $o_\delta(x) \le o_\gamma(x)$.

A proof of (2) is similar: to obtain equality instead of inequality, we use the fact that if $\delta = \gamma_z$ for some $z \in \mathrm{im}(\gamma)$, then $x\delta^{-1} = x\gamma^{-1}$ for every $x \in \mathrm{span}(\delta)$. ☐

Proposition 5.23. *Let $\gamma, \delta \in P(X)$ be connected of type cho with roots x_0 and y_0, respectively. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ if and only if $o(x_0) \le o(y_0)$.*

Proof. Suppose there is an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$. We will prove that $o(x_0) \le o(y_0)$ by transfinite induction on $o(x_0)$. Let $o(x_0) = 1$ (the smallest possible value of $o(x_0)$ since, by definition, a connected partial transformation is not zero). Then the result is true since $o(y_0)$ must be at least 1.

Let $o(x_0) = \mu > 1$ and suppose that for all connected $\gamma_1, \delta_1 \in P(X)$ of type *cho* with roots z and w, respectively, if $o(z) < \mu$ and $\Gamma(\gamma_1)$ is homomorphic to $\Gamma(\delta_1)$, then $o(z) \le o(w)$. Let $z \in x_0\gamma^{-1}$ and note that $o(z) < \mu$. Let $w_z = z\phi$ and note that $w_z \in y_0\delta^{-1}$. Suppose $z \in \mathrm{im}(\gamma)$. Then $w_z \in \mathrm{im}(\delta)$ since $u \xrightarrow{\gamma} z$ for some u implies $u\phi \xrightarrow{\delta} z\phi = w_z$. Thus γ_z and δ_{w_z} are connected of type *cho* with roots z and w_z, respectively (see Notation 5.21). We define ϕ_z on $\downarrow z$ ($= \mathrm{span}(\gamma_z)$) by $\phi_z = \phi|_{\downarrow z}$. Then ϕ_z is an rp-homomorphism from $\Gamma(\gamma_z)$ to $\Gamma(\delta_{w_z})$ since ϕ is an rp-homomorphism and $z\phi = w_z$. Hence, by the inductive hypothesis, $o_{\gamma_z}(z) \le o_{\delta_{w_z}}(w_z)$. By Lemma 5.22, $o_{\gamma_z}(z) = o(z)$ and $o_{\delta_{w_z}}(w_z) = o(w_z)$, and so $o(z) \le o(w_z)$. If $z \notin \mathrm{im}(\gamma)$, then $o(z) = 0$, and so $o(z) \le o(w_z)$. We have proved that for every $z \in x\gamma^{-1}$, there is $w_z \in y_0\delta^{-1}$ such that $o(z) \le o(w_z)$. It follows by (5.4) that $o(x_0) \le o(y_0)$.

Conversely, suppose $o(x_0) \le o(y_0)$. We will prove that $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by transfinite induction on $o(x_0)$. Let $o(x_0) = 1$. Then for every $z \in \mathrm{dom}(\gamma)$, we have $z \xrightarrow{\gamma} x_0$. Since $o(y_0) \ge o(x_0) = 1$, there is some $w \in \mathrm{dom}(\delta)$ such that $w \xrightarrow{\delta} y_0$. Define ϕ on $\mathrm{span}(\gamma)$ by: $x_0\phi = y_0$ and $z\phi = w$ for every $z \in \mathrm{dom}(\gamma)$. Then clearly ϕ is an rp-homomorphism from $\Gamma(\gamma)$ to $\Gamma(\delta)$.

Let $o(x_0) = \mu > 1$ and suppose that for all connected $\gamma_1, \delta_1 \in P(X)$ of type *cho* with roots z and w, respectively, if $o(z) < \mu$ and $o(z) \le o(w)$, then $\Gamma(\gamma_1)$ is rp-homomorphic to $\Gamma(\delta_1)$.

Let $z \in x_0\gamma^{-1}$ and note that $o(z) < \mu$. Since $o(y_0) \ge \mu$, there is $w_z \in y_0\delta^{-1}$ such that $o(z) \le o(w_z)$. If $z \in \mathrm{im}(\gamma)$, then γ_z and δ_{w_z} are connected with $o_{\gamma_z}(z) = o(z) \le o(w_z) = o_{\delta_{w_z}}(w_z)$, and so, by the inductive hypothesis, there is an rp-homomorphism ϕ_z from $\Gamma(\gamma_z)$ to $\Gamma(\delta_{w_z})$. If $z \notin \mathrm{im}(\gamma)$ (that is, if $\downarrow z = \{z\}$), we define ϕ_z on $\downarrow z = \{z\}$ by $z\phi_z = w_z$.

Define ϕ on $\mathrm{span}(\gamma)$ by: $x_0\phi = y_0$ and $u\phi = u\phi_z$ if $u \in \downarrow z$ for some $z \in x_0\gamma^{-1}$. Then ϕ is well-defined since the collection $\{\downarrow z\}_{z \in x_0\gamma^{-1}}$ is a partition of $\mathrm{dom}(\gamma)$ ($= \mathrm{span}(\gamma) - \{x_0\}$). Suppose $u \xrightarrow{\gamma} v$. If $v \in \downarrow z$ for some $z \in x_0\gamma^{-1}$, then $u \in \downarrow z$ as well, and so $u\phi = u\phi_z \xrightarrow{\delta} v\phi_z = v\phi$. If $v = x_0$, then $u = z \in x_0\gamma^{-1}$, and so $u\phi = z\phi = z\phi_z = w_z \xrightarrow{\delta} y_0 = x_0\phi = v\phi$. Hence, since $x_0\phi = y_0$ and x_0 is the unique terminal vertex of $\Gamma(\gamma)$, ϕ is an rp-homomorphism from $\Gamma(\gamma)$ to $\Gamma(\delta)$. ☐

Definition 5.24. Let $\langle a_n \rangle_{n \ge 0}$ and $\langle b_n \rangle_{n \ge 0}$ be sequences of ordinals (indexed by nonnegative integers n). We say that $\langle b_n \rangle$ *dominates* $\langle a_n \rangle$ if there is $k \ge 0$ such that

$$b_{k+n} \ge a_n \text{ for every } n \ge 0.$$

Notation 5.25. Let $\gamma \in P(X)$ be connected of type *rro* and let $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ be a maximal right ray in γ. We denote by $\langle \eta_n^\gamma \rangle_{n \ge 0}$ the sequence of ordinals such that

$$\eta_n^\gamma = o_\gamma(x_n) \text{ for every } n \ge 0.$$

For example, for γ from Example 5.18 and the right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ in γ, the sequence $\langle \eta_n^\gamma \rangle$ is $\langle 0,\, 1,\, \omega,\, \omega+1,\, \omega+2,\, \omega+3, \ldots\rangle$.

Proposition 5.26. *Let $\gamma, \delta \in P(X)$ be connected of type rro. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ if and only if there are maximal right rays η in γ and ξ in δ such that $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$.*

Proof. Suppose there is an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$. Select a maximal right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ in γ (possible by Proposition 5.8.) Then $x_0\phi \xrightarrow{\delta} x_1\phi \xrightarrow{\delta} x_2\phi \xrightarrow{\delta} \cdots$, and so, since δ does not have any double rays, there is $w \in \mathrm{dom}(\delta) - \mathrm{im}(\delta)$ such that $w\delta^k = x_0\phi$ for some $k \geq 0$. Thus

$$\xi = [y_0 = w \;\; y_1 = w\delta \;\ldots\; y_{k-1} = w\delta^{k-1} \;\; y_k = w\delta^k = x_0\phi \;\; y_{k+1} = x_1\phi \;\; y_{k+2} = x_2\phi \;\ldots\rangle$$

is a maximal right ray in δ. For every $n \geq 0$, the mapping $\phi|_{\downarrow x_n}$ is an rp-homomorphism from $\Gamma(\gamma_{x_n})$ to $\Gamma(\delta_{y_{k+n}})$ (see Notation 5.21). Thus for every $n \geq 0$, we have $o_{\gamma_{x_n}}(x_n) \leq o_{\delta_{y_{k+n}}}(y_{k+n})$ by Proposition 5.23, and so $o(x_n) \leq o(y_{k+n})$ by Lemma 5.22. Hence $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$.

Conversely, suppose there are maximal right rays $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ in γ and $\xi = [y_0\, y_1\, y_2 \ldots\rangle$ in δ such that $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$, that is, there is $k \geq 0$ such that $\xi_{k+n}^\delta \geq \eta_n^\gamma$ for every $n \geq 0$. We define a collection $\{B_n\}_{n \geq 0}$ of subsets of $\mathrm{span}(\gamma)$ by

$$B_0 = \{x_0\}, \;\; B_n = \downarrow x_n - \downarrow x_{n-1} \text{ for } n \geq 1.$$

Since γ is connected, $\{B_n\}_{n \geq 0}$ is a partition of $\mathrm{span}(\gamma)$.

We will now define an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$ by defining ϕ on B_n for every $n \geq 0$. First, we set $x_0\phi = y_k$. Let $n \geq 1$. If $B_n = \{x_n\}$, we set $x_n\phi = y_{k+n}$. Suppose $|B_n| \geq 2$. Let $\gamma_n = \gamma|_{B_n - \{x_n\}}$ and $\delta_n = \delta_{y_{k+n}}$. Then γ_n and δ_n are connected of type *cho* with roots x_n and y_{k+n}, respectively. By Lemma 5.22,

$$o_{\gamma_n}(x_n) \leq o_\gamma(x_n) = \eta_n^\gamma \leq \xi_{k+n}^\delta = o_\delta(y_{k+n}) = o_{\delta_n}(y_{k+n}).$$

Thus, by Proposition 5.23, there is an rp-homomorphism ϕ_n from $\Gamma(\gamma_n)$ to $\Gamma(\delta_n)$. Note that $x_n\phi_n = y_{k+n}$. We define ϕ on B_n by $x\phi = x\phi_n$.

Suppose $x \xrightarrow{\gamma} z$. Then $z \in B_n$ for some $n \geq 0$. If $x \in B_n$, then

$$x\phi = x\phi_n \xrightarrow{\delta} z\phi_n = z\phi$$

since ϕ_n is an rp-homomorphism from $\Gamma(\gamma_n)$ to $\Gamma(\delta_n)$. If $x \notin B_n$, then we must have $x = x_{n-1}$ and $z = x_n$, and so

$$x\phi = x_{n-1}\phi = y_{k+n-1} \xrightarrow{\delta} y_{k+n} = x_n\phi = z\phi.$$

Hence, in all cases, if $x \xrightarrow{\gamma} z$ then $x\phi \xrightarrow{\delta} z\phi$. Thus, since $\Gamma(\gamma)$ does not have any terminal vertices, ϕ is an rp-homomorphism from $\Gamma(\gamma)$ to $\Gamma(\delta)$. $\square$

Remark 5.27. Let $S \leq P(X)$ be constant rich, let $\alpha, \beta \in S$ with $\alpha \neq 0$, and let $\phi \in S^1$. It follows from Lemma 4.4(2) and Lemma 4.6 that if ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$, then $\beta \neq 0$.

6 Conjugacy in $T(X)$

In this section we characterize the conjugacy $\sim_c$ in the semigroup $T(X)$ of full transformations on any nonempty set X (finite or infinite).

Proposition 6.1. *Let $S \leq P(X)$ such that S is constant rich or $S \leq T(X)$, and let $\alpha, \beta \in S$ with $\alpha \neq 0$. Then there is an rp-homomorphism $\phi \in S^1$ from $\Gamma(\alpha)$ to $\Gamma(\beta)$ with* $\mathrm{dom}(\phi) = \mathrm{span}(\alpha)$ *if and only if*

(a) *For every connected component γ of α, there exist a connected component δ of β and an rp-homomorphism $\phi_\gamma \in P(X)$ from $\Gamma(\gamma)$ to $\Gamma(\delta)$ with* $\mathrm{dom}(\phi_\gamma) = \mathrm{span}(\gamma)$*; and*

(b) $\bigsqcup_{\gamma\in C}\phi_\gamma \in S^1$, *where C is the collection of connected components of α.*

Proof. Suppose there is an rp-homomorphism $\phi \in S^1$ from $\Gamma(\alpha)$ to $\Gamma(\beta)$ such that $\mathrm{dom}(\phi) = \mathrm{span}(\alpha)$. Let γ be a connected component of α and let $x \in \mathrm{span}(\gamma)$. Then, by Proposition 5.5, $x\phi \in \delta$ for some connected component δ of β. (Note that $\beta \neq \emptyset$ by Remark 5.27.) We claim that $(\mathrm{span}(\gamma))\phi \subseteq \mathrm{span}(\delta)$. Let $z \in \mathrm{span}(\gamma)$. Since γ is connected, $x\alpha^k = x\gamma^k = z\gamma^m = z\alpha^m \neq \diamond$ for some integers $k, m \geq 0$. By Lemma 4.6, we have $\alpha\phi = \phi\beta$, and so $(z\phi)\beta^m = (z\alpha^m)\phi = (x\alpha^k)\phi = (x\phi)\beta^k \neq \diamond$, which implies that $z\phi$ and $x\phi$ are in the span of the same connected component of β, that is, $z\phi \in \mathrm{span}(\delta)$. The claim has been proved. Let $\phi_\gamma = \phi|_{\mathrm{span}(\gamma)}$. Then ϕ_γ is an rp-homomorphism from $\Gamma(\gamma)$ to $\Gamma(\delta)$ (by the claim and the fact that ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$), $\mathrm{dom}(\phi_\gamma) = \mathrm{span}(\gamma)$ (by the definition of ϕ_γ), and $\bigsqcup_{\gamma\in C}\phi_\gamma = \phi \in S^1$ (by the definition of ϕ_γ and the fact that $\mathrm{dom}(\phi) = \mathrm{span}(\alpha)$).

Conversely, suppose that (a) and (b) are satisfied. Let $\phi = \bigsqcup_{\gamma\in C}\phi_\gamma$. Note that ϕ is well defined since ϕ_γ and $\phi_{\gamma'}$ are disjoint if $\gamma \neq \gamma'$. Suppose $y \xrightarrow{\alpha} z$. Then $y, z \in \mathrm{span}(\gamma)$ for some connected component γ of α. Thus $y, z \in \mathrm{dom}(\phi_\gamma)$ and $y\phi = y\phi_\gamma \xrightarrow{\delta} z\phi_\gamma = z\phi$, implying $y\phi \xrightarrow{\beta} z\phi$. Suppose y is a terminal vertex in $\Gamma(\alpha)$ and $y \in \mathrm{dom}(\phi)$. Then, there is a unique connected component γ of α such that y is a terminal vertex in $\Gamma(\gamma)$. Then $y\phi = y\phi_\gamma$ is a terminal vertex in $\Gamma(\delta)$, and so a terminal vertex in $\Gamma(\beta)$. Hence ϕ is an rp-homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Moreover, $\mathrm{dom}(\phi) = \mathrm{span}(\alpha)$ (by the definition of ϕ) and $\phi \in S^1$ (by (b)). ◻

Definition 6.2. Let M be a nonempty subset of the set $\mathbb{Z}_+$ of positive integers. Then M is partially ordered by the relation $\mid$ (divides). Order the elements of M according to the usual "less than" relation: $m_1 < m_2 < m_3 < \ldots$. We define a subset $\mathrm{sac}(M)$ of M as follows: for every integer n, $1 \leq n < |M| + 1$,

$$m_n \in \mathrm{sac}(M) \Leftrightarrow (\forall_{i<n}) m_n \text{ is not a multiple of } m_i.$$

The set $\mathrm{sac}(M)$ is a maximal antichain of the poset $(M, \mid)$. We will call $\mathrm{sac}(M)$ the *standard antichain of M*.

For example, if $M = \{4, 6, 8, 10, 18\}$ then $\mathrm{sac}(M) = \{4, 6, 10\}$; if $M = \{1, 2, 4, 8, 16, 32, \ldots\}$ then $\mathrm{sac}(M) = \{1\}$.

Definition 6.3. Let $\alpha \in P(X)$ such that α contains a cycle. Let

$$M = \{n \in \mathbb{Z}_+ : (\exists_{x\in\mathrm{dom}(\alpha)})\ x\alpha^n = x \text{ and } x\alpha^i \neq x \text{ for every } i, 1 \leq i < n\}.$$

Note that M is the set of the lengths of cycles in α. The standard antichain of $(M, \mid)$ will be called the *cycle set* of α and denoted by $\mathrm{cs}(\alpha)$. We agree that $\mathrm{cs}(\alpha) = \emptyset$ if α has no cycles.

Lemma 6.4. *Let $\alpha, \beta \in P(X)$ be such that $\Gamma(\alpha)$ is rp-homomorphic to $\Gamma(\beta)$. If α has a cycle of length k, then β has a cycle of length m such that $m \mid k$.*

Proof. It follows immediately from Propositions 5.12 and 6.1. ◻

Theorem 6.5. *Let $\alpha, \beta \in T(X)$. Then $\alpha \sim_c \beta$ in $T(X)$ if and only if exactly one of the following conditions is satisfied:*

(1) *Both α and β have a cycle and* $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$.

(2) *Both α and β have a double ray but no cycles.*

(3) *All connected components of both α and β have type rro and:*

   (a) *For every connected component γ of α, there is a connected component δ of β such that $\langle \xi^\delta_n \rangle$ dominates $\langle \eta^\gamma_n \rangle$ for some maximal right ray η in γ and some maximal right ray ξ in δ, and*

(b) *For every connected component δ of β, there is a connected component γ of α such that $\langle \eta_n^{\gamma} \rangle$ dominates $\langle \xi_n^{\delta} \rangle$ for some maximal right ray ξ in δ and some maximal right ray η in γ.*

Proof. Suppose $\alpha \sim_c \beta$. Then, by Corollary 4.8, $\Gamma(\alpha)$ and $\Gamma(\beta)$ are homomorphic to each other. Let ϕ be a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Suppose α has a cycle. Then, by Lemma 6.4, β also has a cycle. Let $n \in \mathrm{cs}(\alpha)$. Then α has a cycle of length n, and so β has a cycle of length m such that $m \,|\, n$. By the definition of $\mathrm{cs}(\beta)$, there is $m_1 \in \mathrm{cs}(\beta)$ such that $m_1 \,|\, m$. Thus β has a cycle of length m_1, and so α has a cycle of length n_1 such that $n_1 \,|\, m_1$, so $n_1 \,|\, m_1 \,|\, m \,|\, n$. Since $\mathrm{cs}(\alpha)$ is an antichain, $n_1 \,|\, n$ and $n \in \mathrm{cs}(\alpha)$ implies $n_1 = n$. Thus $n = m_1$, and so $n \in \mathrm{cs}(\beta)$. We have proved that $\mathrm{cs}(\alpha) \subseteq \mathrm{cs}(\beta)$. By symmetry, $\mathrm{cs}(\beta) \subseteq \mathrm{cs}(\alpha)$, and so $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$.

Suppose α has a double ray, say $\langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ but no cycles. Then β does not have a cycle either by Lemma 6.4, and

$$\dots \xrightarrow{\beta} x_{-1}\phi \xrightarrow{\beta} x_0\phi \xrightarrow{\beta} x_1\phi \xrightarrow{\beta} \dots,$$

where ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. The elements $\dots, x_{-1}\phi, x_0\phi, x_1\phi, \dots$ are pairwise disjoint (since otherwise β would have a cycle), and so $\langle \dots x_{-1}\phi\, x_0\phi\, x_1\phi \dots \rangle$ is a double ray in β.

Suppose that all connected components of α have type *rro*. Then, by the foregoing argument, β cannot have a cycle or a double ray, and so all components of β also have type *rro*. Let γ be a connected component of α and select any maximal right ray η of γ. By Proposition 6.1, there is a connected component δ of β such that $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$. But then, by Proposition 5.26, there is a maximal right ray ξ in δ such that $\langle \xi_n^{\delta} \rangle$ dominates $\langle \eta_n^{\gamma} \rangle$. By symmetry, if δ is a connected component of β and ξ is a maximal right ray in δ, then there is a connected component γ of α and a maximal right ray η in γ such that $\langle \eta_n^{\gamma} \rangle$ dominates $\langle \xi_n^{\delta} \rangle$.

We have proved that one of the conditions (1)–(3) must hold (since $\alpha \in T(X)$ must have a cycle or a double ray or each connected component of α must be of type *rro*). But these conditions are mutually exclusive by Proposition 5.8, and so exactly one of them holds.

Conversely, suppose exactly one of the conditions (1)–(3) holds. Let γ be a connected component of α. We will prove that $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ for some connected component δ of β.

Suppose (1) holds. Suppose γ has a cycle of length k. Then, since $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$, β has a cycle ϑ of length m such that $m \,|\, k$. Let δ be the connected component of β containing ϑ. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Proposition 5.12. Suppose γ has a double chain or γ has type *rro*. Select a connected component δ of β containing a cycle. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemma 5.13.

Suppose (2) holds. Then γ has a double ray or γ has type *rro*. Select a connected component δ of β containing a double ray. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemma 5.14.

Suppose (3) holds. Then there is a connected component δ in β such that $\langle \xi_n^{\delta} \rangle$ dominates $\langle \eta_n^{\gamma} \rangle$ for some maximal right ray η in γ and some maximal right ray ξ in δ. But then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Proposition 5.26.

We have proved that for every connected component γ of α, there exist a connected component δ of β and an rp-homomorphism $\phi_\gamma \in P(X)$ from $\Gamma(\gamma)$ to $\Gamma(\delta)$. We may assume that for every $\gamma \in C(\alpha)$, $\mathrm{dom}(\phi_\gamma) = \mathrm{span}(\gamma)$ (if $\mathrm{dom}(\phi_\gamma) \neq \mathrm{span}(\gamma)$, replace ϕ_γ with $\phi_\gamma|_{\mathrm{span}(\gamma)}$). Moreover, $\mathrm{dom}(\bigsqcup_{\gamma \in C(\alpha)} \phi_\gamma) = \cup_{\gamma \in C(\alpha)} \mathrm{span}(\gamma) = \mathrm{span}(\alpha) = X$ (since $\alpha \in T(X)$). Hence $\Gamma(\alpha)$ is homomorphic to $\Gamma(\beta)$ by Proposition 6.1. By symmetry, $\Gamma(\beta)$ is homomorphic to $\Gamma(\alpha)$, and so $\alpha \sim_c \beta$ by Corollary 4.8. $\square$

Example 6.6. Let $X = \{x_0, x_1, x_2, \dots, y_1, y_2, y_3, \dots\}$ and consider

$$\alpha = [x_0\, x_1\, x_2\, x_3 \dots \rangle,$$
$$\beta = [x_0\, x_1\, x_2\, x_3 \dots \rangle \sqcup [y_1\, y_2\, x_1] \sqcup [y_3\, y_4\, y_5\, y_6\, x_2] \sqcup [y_7\, y_8\, y_9\, y_{10}\, y_{11}\, y_{12}\, x_3] \sqcup \cdots$$

in $P(X)$ (see Figure 6.1). We will argue that α and β are not conjugate. Both α and β are connected of type *rro*. The only maximal right ray in α is $\eta = [x_0\, x_1\, x_2\, x_3 \ldots\rangle$ with $\langle \eta_n^\gamma \rangle = \langle n \rangle$ (where $\gamma = \alpha$). If α and β were conjugate, then $\langle \eta_n \rangle$ would dominate $\langle \xi_n^\delta \rangle$ (where $\delta = \beta$) for some maximal right ray ξ in β, and so for all maximal right rays ξ in β (see Lemma 6.16 below). The right chain $\xi = [x_0\, x_1\, x_2\, x_3 \ldots\rangle$ is a maximal right chain in β with $\langle \xi_n^\delta \rangle = \langle 2n \rangle$. It is clear that the sequence $\langle n \rangle$ does not dominate the sequence $\langle 2n \rangle$. Hence, by Theorem 6.5, α and β are not conjugate.

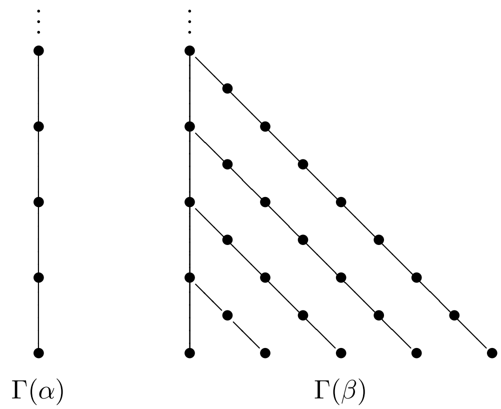

Figure 6.1: The digraphs of α and β from Example 6.6.

If X is a finite set, then every $\alpha \in T(X)$ has a cycle. Hence, Theorem 6.5 gives us the following corollary.

Corollary 6.7. *Let X be finite, and let $\alpha, \beta \in T(X)$. Then $\alpha \sim_c \beta$ if and only if $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$.*

Using Theorem 6.5, we will count the conjugacy classes in $T(X)$ for an infinite set X (Theorem 6.17). We will use the aleph notation for the infinite cardinals, that is, for an ordinal ε, we will write $\aleph_\varepsilon$ for the cardinal indexed by ε. If $\aleph_\varepsilon$ is viewed as an ordinal, we will consistently write ω_ε. This is important because we will need to distinguish between the ordinal and cardinal arithmetic. For example, $\omega_0 < \omega_0 + 1$ (ordinal arithmetic) but $\aleph_0 = \aleph_0 + 1$ (cardinal arithmetic). It will be always clear from the context which arithmetic is used.

A cardinal $\aleph_\varepsilon$ is called *singular* if there is a limit ordinal $\vartheta < \omega_\varepsilon$ and there is an increasing transfinite sequence $\langle \lambda_\nu \rangle_{\nu < \vartheta}$ of ordinals $\lambda_\nu < \omega_\varepsilon$ such that $\omega_\varepsilon = \sup\{\lambda_\nu : \nu < \vartheta\}$ [27, page 160, Definition 2.1]. If $\aleph_\varepsilon$ is not singular, then it is called *regular*.

For any cardinal $\aleph_\varepsilon$, the cardinal $\aleph_{\varepsilon+1}$ is called the *successor* cardinal of $\aleph_\varepsilon$. Every successor cardinal is regular [27, page 162, Theorem 2.4]. The following lemma follows immediately from this fact and the definition of a regular cardinal.

Lemma 6.8. *Let $\aleph_{\varepsilon+1}$ be a successor cardinal and let A be a set of ordinals such that $|A| < \aleph_{\varepsilon+1}$ and $\lambda < \omega_{\varepsilon+1}$ for every $\lambda \in A$. Then $\sup\{\lambda : \lambda \in A\} < \omega_{\varepsilon+1}$.*

To prove the counting theorem, we need a series of lemmas.

Lemma 6.9. *Let $|X| = \aleph_\varepsilon$ and let $\gamma \in P(X)$ be of type cho with root x_0. Then $o(x_0) < \omega_{\varepsilon+1}$.*

Proof. Suppose to the contrary that $o(x_0) \geq \omega_{\varepsilon+1}$. Let $\nu = \sup\{o(y) : y \in x_0\gamma^{-1}\}$. Since $|X| = \aleph_\varepsilon$, we have $|x_0\gamma^{-1}| \leq \aleph_\varepsilon < \aleph_{\varepsilon+1}$. If $o(y) < \omega_{\varepsilon+1}$ for every $y \in x_0\gamma^{-1}$, then $\nu < \omega_{\varepsilon+1}$ by Lemma 6.8, and so $o(x_0) < \omega_{\varepsilon+1}$ by (5.4), which is a contradiction.

Thus, there is $y_1 \in x_0\gamma^{-1}$ such that $o(y_1) \geq \omega_{\varepsilon+1}$. Applying the foregoing argument to y_1, we obtain $y_2 \in y_1\gamma^{-1}$ such that $o(y_2) \geq \omega_{\varepsilon+1}$. Note that $y_2 \xrightarrow{\gamma} y_1 \xrightarrow{\gamma} x_0$. Continuing this way, we obtain a sequence $y_1, y_2, y_3, y_4, \ldots$ of elements of $\mathrm{dom}(\gamma)$ such that $o(y_i) \geq \omega_{\varepsilon+1}$ for each i and

$$\cdots \xrightarrow{\gamma} y_4 \xrightarrow{\gamma} y_3 \xrightarrow{\gamma} y_2 \xrightarrow{\gamma} y_1 \xrightarrow{\gamma} x_0.$$

But this is a contradiction since γ does not have any left rays. Hence $o(x_0) < \omega_{\varepsilon+1}$. $\square$

Lemma 6.10. *Let $|X| = \aleph_\varepsilon$. Then for every nonzero ordinal $\mu < \omega_{\varepsilon+1}$, there is $\gamma \in P(X)$ of type cho with root x_0 such that $o(x_0) = \mu$.*

Proof. Let $0 < \mu < \omega_{\varepsilon+1}$. We proceed by transfinite induction. The result is clearly true if $\mu = 1$. Let $\mu > 1$ and suppose that the result is true for every ordinal λ such that $0 < \lambda < \mu$.

Fix $x_0 \in X$, let $X_0 = X - \{x_0\}$, and note that $|X_0| = \aleph_\varepsilon$. Since $\mu < \omega_{\varepsilon+1}$, we have $|\mu| \leq \aleph_\varepsilon$. Thus, since $\aleph_\varepsilon \cdot \aleph_\varepsilon = \aleph_\varepsilon$ and $\mu = \{\lambda : \lambda \text{ is an ordinal such that } \lambda < \mu\}$, there is a collection $\{X_\lambda\}_{0<\lambda<\mu}$ of pairwise disjoint subsets of X_0 such that $|X_\lambda| = \aleph_\varepsilon$ for every λ.

Let $0 < \lambda < \mu$. By the inductive hypothesis, there is $\gamma_\lambda \in P(X_\lambda)$ of type *cho* with root x_λ such that $o(x_\lambda) = \lambda$. We define $\gamma \in P(X)$ as follows. Set $\mathrm{dom}(\gamma) = \bigcup_{0<\lambda<\mu} \mathrm{span}(\gamma_\lambda)$. For every $x \in \mathrm{dom}(\gamma)$, define

$$x\gamma = \begin{cases} x\gamma_\lambda & \text{if } x \in \mathrm{dom}(\gamma_\lambda), \\ x_0 & \text{if } x = x_\lambda. \end{cases}$$

Then γ is of type *cho*, x_0 is the root of γ, and $x_0\gamma^{-1} = \{x_\lambda : 0 < \lambda < \mu\}$. Let $\nu = \sup\{o(y) : y \in x_0\gamma^{-1}\}$. Then

$$\nu = \sup\{o(x_\lambda) : 0 < \lambda < \mu\} = \sup\{\lambda : 0 < \lambda < \mu\},$$

where the last equality is true since $o(x_\lambda) = \lambda$ for every nonzero $\lambda < \mu$. Hence, either $\mu = \nu$ (if μ is a limit ordinal) or $\mu = \nu + 1$ (if $\nu = \lambda$ for some nonzero $\lambda < \mu$). It follows by (5.4) that $o(x_0) = \mu$. $\square$

Lemma 6.11. *Let $|X| = \aleph_\varepsilon$ and let $\langle a_n \rangle$ be an increasing sequence of ordinals $a_n < \omega_{\varepsilon+1}$ such that $a_0 = 0$. Then there is $\gamma \in T(X)$ of type rro with a maximal right ray η such that $\langle \eta_n^\gamma \rangle = \langle a_n \rangle$.*

Proof. Since $|X| = \aleph_\varepsilon$, there is a collection $\{X_n\}_{n\geq 0}$ of pairwise disjoint subsets of X such that $X_0 = \{x_0\}$ and $|X_n| = \aleph_\varepsilon$ for every $n \geq 1$. Let $n \geq 1$. By Lemma 6.10, there is $\gamma_n \in P(X_n)$ of type *cho* with root x_n such that $o_{\gamma_n}(x_n) = a_n$. Define $\gamma \in T(X)$ by

$$x\gamma = \begin{cases} x\gamma_n & \text{if } x \in \mathrm{dom}(\gamma_n), \\ x_{n+1} & \text{if } x = x_n, \\ x_1 & \text{for any other } x. \end{cases}$$

Then γ is of type *rro* (since every γ_n is of type *cho*). By the definition of γ, we have that $o_\gamma(x_0) = 0$ and $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ is a maximal right ray in γ. We have already noticed that $o_\gamma(x_0) = 0 = a_0$. We will prove by induction on n that $o_\gamma(x_n) = a_n$ for every $n \geq 1$. Let $n = 1$. Then, since $a_1 \geq 1$,

$$o_\gamma(x_1) = \max\{1, o_{\gamma_1}(x_1)\} = \max\{1, a_1\} = a_1.$$

Let $n \geq 1$ and suppose $o_\gamma(x_n) = a_n$. Then

$$o_\gamma(x_{n+1}) = \max\{o_\gamma(x_n) + 1, o_{\gamma_{n+1}}(x_{n+1})\} = \max\{a_n + 1, a_{n+1}\} = a_{n+1},$$

where the last equality is true since $\langle a_n \rangle$ is increasing, and so $a_{n+1} > a_n$. This concludes the inductive argument. Thus $\eta_n^\gamma = o_\gamma(x_n) = a_n$ for every $n \geq 0$, which completes the proof. $\square$

Lemma 6.12. *Let $\aleph_{\varepsilon+1}$ be a successor cardinal. Then there is a collection $\{\langle a_n^\mu \rangle\}_{\mu<\omega_{\varepsilon+1}}$ of increasing sequences $\langle a_n^\mu \rangle$ of ordinals $a_n^\mu < \omega_{\varepsilon+1}$ such that for all ordinals $\mu, \lambda < \omega_{\varepsilon+1}$, $a_0^\mu = 0$ and if $\lambda < \mu$ then $a_m^\lambda < a_n^\mu$ for all $m, n \geq 1$.*

Proof. We construct such a collection by transfinite recursion. We define $\langle a_n^0 \rangle = \langle 0, 1, 2, 3, \ldots \rangle$. Let μ be an ordinal such that $0 < \mu < \omega_{\varepsilon+1}$ and suppose $\langle a_n^\lambda \rangle$ has been defined for every ordinal $\lambda < \mu$. Let $\tau = \sup\{a_n^\lambda : \lambda < \mu \text{ and } n \geq 0\}$. Let $A = \{a_n^\lambda : \lambda < \mu \text{ and } n \geq 0\}$. Then $|A| = |\mu| \cdot \aleph_0 < \aleph_{\varepsilon+1}$, and so $\tau < \omega_{\varepsilon+1}$ by Lemma 6.8. Define $\langle a_n^\mu \rangle = \langle 0, \tau+1, \tau+2, \tau+3, \ldots \rangle$ and note that $\langle a_n^\mu \rangle$ is an increasing sequence of ordinals $a_n^\mu < \omega_{\varepsilon+1}$ with $a_0^\mu = 0$. The construction has been completed. It is clear by the construction that $a_m^\lambda < a_n^\mu$ for all $\lambda, \mu < \omega_{\varepsilon+1}$ with $\lambda < \mu$ and all $m, n \geq 1$. □

Remark 6.13. Let $\{\langle a_n^\mu \rangle\}_{\mu < \omega_{\varepsilon+1}}$ be a collection from Lemma 6.12. Then it is clear that for all ordinals $\lambda, \mu < \omega_{\varepsilon+1}$, if $\lambda < \mu$ then $\langle a_n^\lambda \rangle$ does not dominate $\langle a_n^\mu \rangle$.

Definition 6.14. Let $\aleph_{\varepsilon+1}$ be a successor cardinal. Denote by $\mathrm{IS}_{\omega_{\varepsilon+1}}$ the set of all increasing sequences $\langle a_n \rangle$ of ordinals $a_n < \omega_{\varepsilon+1}$ such that $a_0 = 0$. Define a relation $\approx$ on $\mathrm{IS}_{\omega_{\varepsilon+1}}$ by

$$\langle a_n \rangle \approx \langle b_n \rangle \text{ if } \langle b_n \rangle \text{ dominates } \langle a_n \rangle \text{ and } \langle a_n \rangle \text{ dominates } \langle b_n \rangle.$$

It is straightforward to show that $\approx$ is an equivalence relation on $\mathrm{IS}_{\omega_{\varepsilon+1}}$. We denote by $[\langle a_n \rangle]_\approx$ the equivalence class of $\langle a_n \rangle$, and by $\mathrm{IS}_{\omega_{\varepsilon+1}}^\approx$ the set of all equivalence classes of $\approx$.

Lemma 6.15. $|\mathrm{IS}_{\omega_{\varepsilon+1}}| = \aleph_{\varepsilon+1}^{\aleph_0}$ *and* $\aleph_{\varepsilon+1} \leq |\mathrm{IS}_{\omega_{\varepsilon+1}}^\approx| \leq \aleph_{\varepsilon+1}^{\aleph_0}$.

Proof. Denote by $S_{\omega_{\varepsilon+1}}$ the set of all sequences $\langle s_n \rangle$ of ordinals $s_n < \omega_{\varepsilon+1}$. Then $S_{\omega_{\varepsilon+1}}$ is the set of all functions from $\mathbb{N}$ to $\omega_{\varepsilon+1}$, and so $|S_{\omega_{\varepsilon+1}}| = |\omega_{\varepsilon+1}|^{|\mathbb{N}|} = \aleph_{\varepsilon+1}^{\aleph_0}$. Since $\mathrm{IS}_{\omega_{\varepsilon+1}}$ is a subset of $S_{\omega_{\varepsilon+1}}$, we have $|\mathrm{IS}_{\omega_{\varepsilon+1}}| \leq \aleph_{\varepsilon+1}^{\aleph_0}$. Let $S_{\omega_{\varepsilon+1}}^0$ be the subset of $S_{\omega_{\varepsilon+1}}$ consisting of all sequences $\langle s_n \rangle$ such that $s_n > 0$ for all $n \geq 0$. Then $|S_{\omega_{\varepsilon+1}}^0| = |S_{\omega_{\varepsilon+1}}| = \aleph_{\varepsilon+1}^{\aleph_0}$. Define a function $f : S_{\omega_{\varepsilon+1}}^0 \to \mathrm{IS}_{\omega_{\varepsilon+1}}$ by $\langle s_n \rangle f = \langle a_n \rangle$, where

$$a_0 = 0 \text{ and } a_{n+1} = a_n + s_n \text{ for all } n \geq 0.$$

Then f is injective (since for all ordinals $\mu, \lambda_1, \lambda_2$, if $\mu + \lambda_1 = \mu + \lambda_2$ then $\lambda_1 = \lambda_2$ [27, page 120, Lemma 5.4]), and so $|\mathrm{IS}_{\omega_{\varepsilon+1}}| \geq |S_{\omega_{\varepsilon+1}}^0| = \aleph_{\varepsilon+1}^{\aleph_0}$. We have proved that $|\mathrm{IS}_{\omega_{\varepsilon+1}}| = \aleph_{\varepsilon+1}^{\aleph_0}$.

We have $|\mathrm{IS}_{\omega_{\varepsilon+1}}^\approx| \leq |\mathrm{IS}_{\omega_{\varepsilon+1}}| = \aleph_{\varepsilon+1}^{\aleph_0}$. Let $\{\langle a_n^\mu \rangle\}_{\mu < \omega_{\varepsilon+1}}$ be a collection of sequences from Lemma 6.12. Then for all ordinals $\lambda, \mu < \omega_{\varepsilon+1}$, $\langle a_n^\mu \rangle \in \mathrm{IS}_{\omega_{\varepsilon+1}}$ and if $\lambda < \mu$ then $\langle a_n^\lambda \rangle$ does not dominate $\langle a_n^\mu \rangle$ (see Remark 6.13). It follows that any two different sequences from the collection $\{\langle a_n^\mu \rangle\}_{\mu < \omega_{\varepsilon+1}}$ are in different equivalence classes of $\approx$. Since there are $\aleph_{\varepsilon+1}$ sequences in the collection, it follows that $|\mathrm{IS}_{\omega_{\varepsilon+1}}^\approx| \geq \aleph_{\varepsilon+1}$. This concludes the proof. □

Lemma 6.16. *Let $\gamma, \delta \in P(X)$ be of type rro. Let η be a maximal right ray in γ and ξ be a maximal right ray in δ such that $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$. Then for every maximal right ray η_1 in γ and every maximal right ray ξ_1 in δ $\langle (\xi_1)_n^\delta \rangle$ dominates $\langle (\eta_1)_n^\gamma \rangle$.*

Proof. Since $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$, there is an integer $k \geq 0$ such that

$$\xi_{k+n}^\delta \geq \eta_n^\gamma \text{ for every } n \geq 0.$$

Let $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ and $\xi = [y_0\, y_1\, y_2 \ldots\rangle$. Let $\eta_1 = [w_0\, w_1\, w_2 \ldots\rangle$ and $\xi_1 = [z_0\, z_1\, z_2 \ldots\rangle$ be arbitrary maximal right rays in γ and δ, respectively. Since γ and δ are connected, there are integers $l, q, m, p \geq 0$ such that $x_l = x_0\gamma^l = w_0\gamma^q = w_q$ and $y_m = y_0\delta^m = z_0\delta^p = z_p$. We may assume that $m \geq k$. Then for every $n \geq 0$,

$$\begin{aligned}(\xi_1)_{(p+l)+n}^\delta &= o_\delta(z_{p+(l+n)}) = o_\delta(y_{m+(l+n)}) \geq o_\delta(y_{k+(l+n)}) = \xi_{k+(l+n)}^\delta \geq \eta_{l+n}^\gamma, \text{ and} \\ \eta_{l+n}^\gamma &= o_\gamma(x_{l+n}) = o_\gamma(w_{q+n}) \geq o_\gamma(w_n) = (\eta_1)_n^\gamma.\end{aligned}$$

Hence $\langle (\xi_1)_n^\delta \rangle$ dominates $\langle (\eta_1)_n^\gamma \rangle$. □

We are now ready to prove the counting theorem. For a set A, we denote by $\mathcal{P}(A)$ the power set of A.

Theorem 6.17. *Let X be an infinite set with $|X| = \aleph_\varepsilon$. Then in $T(X)$ there are:*

(1) *$2^{\aleph_0}$ conjugacy classes consisting of transformations with a cycle, of which $\aleph_0$ have a connected representative;*

(2) *One conjugacy class consisting of transformations with a double ray but not a cycle;*

(3) *$2^{\aleph_\varepsilon}$ conjugacy classes consisting of transformations without a cycle or a double ray, of which at least $\aleph_{\varepsilon+1}$ and at most $\aleph_{\varepsilon+1}^{\aleph_0}$ have a connected representative.*

In total, there are $2^{\aleph_\varepsilon}$ conjugacy classes in $T(X)$, of which at least $\aleph_{\varepsilon+1}$ and at most $\aleph_{\varepsilon+1}^{\aleph_0}$ have a connected representative.

Proof. For $\alpha \in T(X)$, we denote by $[\alpha]_c$ the conjugacy class of α. Let $A = \{[\alpha]_c : \alpha \text{ has a cycle}\}$. Define a function $f : A \to \mathcal{P}(\mathbb{Z}_+)$ by $([\alpha)]_c)f = \mathrm{cs}(\alpha)$. By Theorem 6.5(1), f is well defined and injective. Thus $|A| \leq |\mathcal{P}(\mathbb{Z}_+)| = 2^{\aleph_0}$. Let P be the set of prime positive integers. For any nonempty subset $Q \subseteq P$, let $\{\theta_q\}_{q\in Q}$ be a collection of completely disjoint cycles θ_q such that θ_q has length q for every $q \in Q$. (Such a collection exists since X is infinite.) Define $\beta_{\scriptscriptstyle Q} \in P(X)$ by $\beta_{\scriptscriptstyle Q} = \bigsqcup_{q\in Q} \theta_q$. Now fix $y_0 \in \mathrm{dom}(\beta_{\scriptscriptstyle Q})$ and define $\alpha_{\scriptscriptstyle Q} \in T(X)$ by

$$\alpha_{\scriptscriptstyle Q} = \beta_{\scriptscriptstyle Q} \vee \bigsqcup_{x\notin \mathrm{dom}(\beta_Q)} [x, y_0].$$

Note that $\mathrm{cs}(\alpha_{\scriptscriptstyle Q}) = Q$. For all nonempty subsets $Q_1, Q_2 \subseteq P$ with $Q_1 \neq Q_2$, we have $(\alpha_{\scriptscriptstyle Q_1}, \alpha_{\scriptscriptstyle Q_2}) \notin\, \sim_c$ by Theorem 6.5(1). It follows that $|A| \geq \mathcal{P}(P) = 2^{\aleph_0}$. Hence $|A| = 2^{\aleph_0}$.

Let $A_1 = \{[\gamma]_c : \gamma \text{ has a cycle and } \gamma \text{ is connected}\}$. By Proposition 5.8, if a connected $\gamma \in T(X)$ has a cycle, then the cycle is unique. Fix a subset $X_0 = \{x_0, x_1, x_2, \ldots\}$ of X of cardinality $\aleph_0$. For every integer $n \geq 0$, define a connected $\gamma_n \in T(X)$ by

$$\gamma_n = (x_0\, x_1 \ldots x_n) \vee \bigsqcup_{x\notin X_0} [x, x_0].$$

Then, by Theorem 6.5(1), we have $A_1 = \{[\gamma_0]_c, [\gamma_1]_c, [\gamma_2]_c, \ldots\}$, and so $|A_1| = \aleph_0$. We have proved (1). Statement (2) follows immediately from Theorem 6.5(2).

Let $B = \{[\alpha]_c : \alpha \text{ does not have a cycle or a double ray}\}$ and let B_1 be the subset of B consisting of all conjugacy classes $[\gamma]_c \in B$ such that γ is connected. Note that $B_1 = \{[\gamma]_c : \gamma \text{ is of type } rro\}$. For every $\gamma \in T(X)$ of type *rro*, we fix a maximal right ray η^γ in γ. Define a function $g : B_1 \to \mathrm{IS}^{\approx}_{\omega_{\varepsilon+1}}$ by $([\gamma]_c)g = [\langle \eta^\gamma_n \rangle]_\approx$. Note that $\langle \eta^\gamma_n \rangle \in \mathrm{IS}_{\omega_{\varepsilon+1}}$ by Lemma 6.9. Suppose $[\gamma_1]_c, [\gamma_2]_c \in B_1$ with $[\gamma_1]_c = [\gamma_2]_c$. Then, by Theorem 6.5(3) and Lemma 6.16, the sequences $\langle \eta^{\gamma_1}_n \rangle$ and $\langle \eta^{\gamma_2}_n \rangle$ dominate each other, and so $[\langle \eta^{\gamma_1}_n \rangle]_\approx = [\langle \eta^{\gamma_2}_n \rangle]_\approx$. We have proved that g is well defined. The function g is also injective (by Theorem 6.5(3)) and surjective (by Lemma 6.11). Thus $|B_1| = |\mathrm{IS}^{\approx}_{\omega_{\varepsilon+1}}|$, and so, by Lemma 6.15, $\aleph_{\varepsilon+1} \leq |B_1| \leq \aleph_{\varepsilon+1}^{\aleph_0}$.

We will now prove that $|B| = 2^{\aleph_\varepsilon}$. First, clearly $|B| \leq |T(X)| = \aleph_\varepsilon^{\aleph_\varepsilon} = 2^{\aleph_\varepsilon}$. Since $|B_1| \geq \aleph_{\varepsilon+1}$, there is a collection $\{\gamma_\mu\}_{\mu<\omega_{\varepsilon+1}}$ of transformations $\gamma_\mu \in T(X)$ of type *rro* such that $(\gamma_\mu, \gamma_\lambda) \notin\, \sim_c$ if $\mu \neq \lambda$. Since $|\omega_\varepsilon| = \aleph_\varepsilon$ and $\aleph_\varepsilon \cdot \aleph_\varepsilon = \aleph_\varepsilon$, there is a partition $\{X_\mu\}_{\mu<\omega_\varepsilon}$ of X such that $|X_\mu| = |X| = \aleph_\varepsilon$ for every $\mu < \omega_\varepsilon$. Let $\mu < \omega_\varepsilon$. Since $|X_\mu| = |X|$, there is a bijection $h_\mu : X_\mu \to X$. We can use the bijection h_μ to obtain a "copy" of γ_μ in $T(X_\mu)$: define $\gamma'_\mu \in T(X_\mu)$ by

$$x\gamma'_\mu = y \Leftrightarrow (xh_\mu)\gamma_\mu = yh_\mu \text{ (for all } x, y \in X_\mu).$$

Let $\mu, \lambda < \omega_\varepsilon$ with $\mu \neq \lambda$. Then $(\gamma_\mu, \gamma_\lambda) \notin\, \sim_c$, and so, by Theorem 6.5(3) and Lemma 6.16, $(\langle \eta_n \rangle, \langle \xi_n \rangle) \notin\, \approx$ for every maximal right ray η in γ_μ and every maximal right ray ξ in γ_λ. It follows that

$$(\langle \eta'_n \rangle, \langle \xi'_n \rangle) \notin\, \approx \tag{6.1}$$

for every maximal right ray η' in γ'_μ and every maximal right ray ξ' in γ'_λ. Let K be a nonempty subset of ω_ε. Select $\nu = \nu_K \in K$ and a maximal right ray $\langle x_0\, x_1\, x_2 \ldots \rangle$ in γ'_ν. Define $\beta_K \in P(X)$ by $\beta_K = \bigsqcup_{\mu \in K} \gamma'_\mu$, and then define $\alpha_K \in T(X)$ by

$$\alpha_K = \beta_K \sqcup \bigsqcup_{x \notin \mathrm{dom}(\beta_K)} [x, x_1],$$

and note that α_K does not have a cycle or a double ray. Let K, L be nonempty subsets of ω_ε such that $K \neq L$. We may assume that there is $\mu \in K$ such that $\mu \notin L$. Consider γ'_μ, which is a connected component of α_K. Let γ'_λ be any connected component of α_L. Then, by (6.1), $(\langle \eta'_n \rangle, \langle \xi'_n \rangle) \notin \approx$ for every maximal right ray η' in γ'_μ and every maximal right ray ξ' in γ'_λ. (Note that, by the definition of α_K, this is also true when $\mu = \nu_K$ or $\lambda = \nu_L$.) Thus $(\alpha_K, \alpha_L) \notin \sim_c$ by Theorem 6.5(3). Hence any two different transformations from the collection $\{\alpha_K\}_{\emptyset \neq K \subseteq \omega_\varepsilon}$ are in different equivalence classes of $\sim_c$. Since there are $2^{\aleph_\varepsilon}$ transformations in the collection, it follows that $|B| \geq 2^{\aleph_\varepsilon}$. Thus $|B| = 2^{\aleph_\varepsilon}$, which concludes the proof of (3).

By (1)–(3), there are $2^{\aleph_0} + 1 + 2^{\aleph_\varepsilon} = 2^{\aleph_\varepsilon}$ conjugacy classes in $T(X)$, of which at least $\aleph_0 + 1 + \aleph_{\varepsilon+1} = \aleph_{\varepsilon+1}$ and at most $\aleph_0 + 1 + \aleph_{\varepsilon+1}^{\aleph_0} = \aleph_{\varepsilon+1}^{\aleph_0}$ have a connected representative. $\square$

7 Conjugacy in $P(X)$

A characterization of the conjugacy $\sim_c$ in the monoid $P(X)$ of partial transformations on X is more complicated than that of the conjugacy in $T(X)$ (see Section 6). The reason is that a connected component of $\alpha \in P(X)$ can have a maximal left ray or be of type *cho* (chains only), which is impossible for $\alpha \in T(X)$. This entails several complications. For example, in contrast with $T(X)$, $\alpha, \beta \in P(X)$ with a cycle and $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$ are not necessarily conjugate. Also in contrast with $T(X)$, $\alpha, \beta \in P(X)$ with a double ray may lie in different conjugate classes.

Lemma 7.1. *Let $\gamma, \delta \in P(X)$ be connected. Suppose that δ has a maximal left ray and γ either has a maximal left ray or is of type cho. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$.*

Proof. Let $\langle \ldots\, y_2\, y_1\, y_0]$ be a maximal left ray in δ. Note that y_0 is the root of δ. Let x_0 be the root of γ. We will define an rp-homomorphism ϕ from $\Gamma(\gamma)$ to $\Gamma(\delta)$ such that $\mathrm{dom}(\phi) = \mathrm{span}(\gamma)$ and $\mathrm{im}(\phi) \subseteq \{\ldots, y_2, y_1, y_0\}$. For $x \in \mathrm{span}(\gamma)$, let p_x be the smallest nonnegative integer such that $x\gamma^{p_x} = x_0$ (such p_x exists by Lemma 5.7). Define ϕ on $\mathrm{span}(\gamma)$ by $x\phi = y_{p_x}$. If $x \xrightarrow{\gamma} z$, then $p_z = p_x - 1$, and so

$$x\phi = y_{p_x} \xrightarrow{\delta} y_{p_x - 1} = y_{p_z} = z\phi.$$

Further, the only terminal vertex in $\Gamma(\gamma)$ is x_0 and $x_0\phi = y_0$ (since $p_{x_0} = 0$), which is a terminal vertex in $\Gamma(\delta)$. Hence ϕ is an rp-homomorphism. $\square$

Lemma 7.2. *Let $\gamma, \delta \in P(X)$ be connected such that γ is of type rro. Suppose $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$. Then δ cannot have a maximal left ray or be of type cho.*

Proof. Let ϕ be an rp-homomorphism from $\Gamma(\gamma)$ to $\Gamma(\delta)$. Select a right ray $\langle x_0\, x_1\, x_2 \ldots \rangle$ in γ. Suppose to the contrary that δ has a maximal left ray or is of type *cho*. Let y_0 be the root of δ. By Lemma 5.7, $(x_0\phi)\delta^k = y_0$ for some integer $k \geq 0$. By Lemma 4.6, $\gamma\phi = \phi\delta$, and so

$$(x_0\phi)\delta^{k+1} = (x_0\gamma^{k+1})\phi = x_{k+1}\phi.$$

But $(x_0\phi)\delta^{k+1} = (x_0\phi)\delta^k\delta = y_0\delta = \diamond$, and so $x_{k+1}\phi = \diamond$, which is a contradiction. The result follows. $\square$

Theorem 7.3. *Let $\alpha, \beta \in P(X)$. Then $\alpha \sim_c \beta$ in $P(X)$ if and only if $\alpha = \beta = 0$ or $\alpha, \beta \neq 0$ and the following conditions are satisfied:*

(1) $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$;

(2) *α has a double ray but not a cycle $\Leftrightarrow$ β has a double ray but not a cycle;*

(3a) *If α has a connected component γ of type rro but no cycles or double rays, then β has a connected component δ of type rro but no cycles or double rays, and $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$ for some maximal right rays η in γ and ξ in δ;*

(3b) *If β has a connected component δ of type rro but no cycles or double rays, then α has a connected component γ of type rro but no cycles or double rays, and $\langle \eta_n^\gamma \rangle$ dominates $\langle \xi_n^\delta \rangle$ for some maximal right rays ξ in β and η in γ;*

(4) *α has a maximal left ray $\Leftrightarrow$ β has a maximal left ray;*

(5a) *If α has a connected component γ of type cho with root x_0 but no maximal left rays, then β has a connected component δ of type cho with root y_0 but no maximal left rays, and $o_\gamma(x_0) \le o_\delta(y_0)$;*

(5b) *If β has a connected component δ of type cho with root y_0 but no maximal left rays, then α has a connected component γ of type cho with root x_0 but no maximal left rays, and $o_\delta(y_0) \le o_\gamma(x_0)$.*

Proof. Suppose $\alpha \sim_c \beta$. Then, since $[0]_{\sim_c} = \{0\}$ in every semigroup with 0, either $\alpha = \beta = 0$ or $\alpha, \beta \ne 0$. Suppose $\alpha, \beta \ne 0$. Then, by Theorem 4.7, there is an rp-homomorphism ϕ from $\Gamma(\alpha)$ to $\Gamma(\beta)$. We may assume that $\mathrm{dom}(\phi) = \mathrm{span}(\alpha)$.

Suppose α has a cycle. By the argument identical to that we used in the proof of Theorem 6.5, β also has a cycle and $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$. By symmetry, if β has a cycle, then α also has a cycle and $\mathrm{cs}(\beta) = \mathrm{cs}(\alpha)$. If neither α nor β has a cycle, then $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta) = \emptyset$. We have proved (1).

Suppose α has a double ray but not a cycle. Then, again by the argument identical to that we used in the proof of Theorem 6.5, β has a double ray but not a cycle. The converse is true by symmetry. This proves (2).

Suppose that α has a connected component γ of type *rro* but not a cycle or a double ray. By Proposition 6.1, there is a connected component δ of β such that $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$. By (1) and (2), δ does not have a cycle or a double ray. By Lemma 7.2, δ does not have a maximal left ray and it is not of type *cho*. Hence δ has type *rro*. By Proposition 5.26, there are maximal right rays η in γ and ξ in δ such that $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$. We have proved (3a). Condition (3b) holds by symmetry.

Suppose α has a maximal left ray, say $\langle \dots x_2\, x_1\, x_0]$. Then

$$\dots \xrightarrow{\beta} x_2\phi \xrightarrow{\beta} x_1\phi \xrightarrow{\beta} x_0\phi$$

and $x_0\phi$ is a terminal vertex in $\Gamma(\beta)$, which implies that $\langle \dots x_2\phi\, x_1\phi\, x_0\phi]$ is a maximal left ray in β. The converse is true by symmetry. This proves (4).

Suppose α has a connected component γ of type *cho* with root x_0 but not a maximal left ray. By Proposition 6.1 and its proof, there is a connected component δ of β such that $\phi_\gamma = \phi|_{\mathrm{span}(\gamma)}$ is an rp-homomorphism from $\Gamma(\gamma)$ to $\Gamma(\delta)$. Since x_0 is a terminal vertex in γ, $y_0 = x_0\phi_\gamma$ is a terminal vertex in δ. Since β has no maximal left ray (by (3)), δ is of type *cho* and y_0 is the root of δ. By Proposition 5.23, $o_\gamma(x_0) \le o_\delta(y_0)$. We have proved (5a). Condition (5b) holds by symmetry.

Conversely, if $\alpha = \beta = 0$ then $\alpha \sim_c \beta$. Suppose that $\alpha, \beta \ne 0$ and that (1)–(5b) hold. Let γ be a connected component of α. We will prove that $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ for some connected component δ of β.

Suppose γ has a cycle of length k. Since, by (1), $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$, β has a cycle ϑ of length m such that $m \,|\, k$. Let δ be the connected component of β containing ϑ. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Proposition 5.12.

Suppose γ has a double ray. If some connected component δ of β has a cycle, then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemma 5.13. Suppose β does not have a cycle. Then, by (1) and (2),

both α and β have a double ray but not a cycle. Let δ be a connected component of β containing a double ray. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemma 5.14.

Suppose γ is of type *rro*. If β has some connected component δ with a cycle or a double ray, then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemmas 5.13 and 5.14. Suppose β does not have a cycle or a double ray. Then, by (3a), there is a connected component δ in β of type *rro* such that $\langle \xi_n^\delta \rangle$ dominates $\langle \eta_n^\gamma \rangle$ for some maximal right rays η in γ and ξ in δ. Hence $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Proposition 5.26.

Suppose γ has a maximal left ray. Then, by (4), some connected component δ of β has a maximal left ray. Then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemma 7.1.

Suppose γ is of type *cho* with root x_0. If β has some connected component δ with a maximal left ray, then $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Lemma 7.1. Suppose β does not have a maximal left ray. Then, by (4), α does not have a maximal left ray, and so, by (5a), there is a connected component δ in β of type *cho* with root y_0 such that $o_\gamma(x_0) \leq o_\delta(y_0)$. Hence $\Gamma(\gamma)$ is rp-homomorphic to $\Gamma(\delta)$ by Proposition 5.23.

We have proved that for every connected component γ of α, there exists a connected component δ of β and an rp-homomorphism $\phi_\gamma \in P(X)$ from $\Gamma(\gamma)$ to $\Gamma(\delta)$. We may assume that for every $\gamma \in C(\alpha)$, $\mathrm{dom}(\phi_\gamma) = \mathrm{span}(\gamma)$. Hence $\Gamma(\alpha)$ is rp-homomorphic to $\Gamma(\beta)$ by Proposition 6.1. By symmetry, $\Gamma(\beta)$ is rp-homomorphic to $\Gamma(\alpha)$, and so $\alpha \sim_c \beta$ by Theorem 4.7. $\square$

Example 7.4. Let X be an infinite set containing $x_0, y_1, y_2, y_3, \ldots$ and let $\alpha, \beta \in P(X)$ be the partial transformations whose digraphs are presented in Figure 7.1. Then α is connected of type *cho* with root x_0, and $\beta = \delta_1 \sqcup \delta_2 \sqcup \delta_3 \sqcup \delta_4 \sqcup \cdots$, where δ_i is the chain with root y_i. We have $o_\gamma(x_0) = \omega$, where $\gamma = \alpha$, and for every integer $i \geq 1$, $o_{\delta_i}(y_i) = i$. Hence α and β are not conjugate by (5a) of Theorem 7.3.

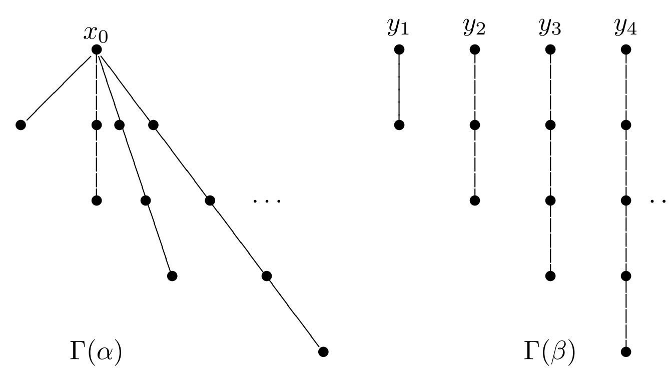

Figure 7.1: The digraphs of α and β from Example 7.4.

Definition 7.5. For $\alpha \in P(X)$, we define

$$s(\alpha) = \sup\{o_\gamma(x_0) : \gamma \text{ is a connected component of } \alpha \text{ of type } cho \text{ with root } x_0\},$$

where we agree that $s(\alpha) = 0$ if α has no connected component of type *cho*.

Suppose $\alpha, \beta \in P(X)$ have a connected component of type *cho* but no cycles or rays. Then, by Theorem 7.3, if $\alpha \sim_c \beta$ then $s(\alpha) = s(\beta)$. However, the converse is not true. Indeed, consider $\alpha, \beta \in P(X)$ from Example 7.4 (see Figure 7.1). Then α is connected of type *cho* with the root of order ω, and β is a join of connected components of type *cho* (chains) whose roots have orders $1, 2, 3, 4, \ldots$. Thus $s(\alpha) = s(\beta) = \omega$, but $(\alpha, \beta) \notin \sim_c$ by (5a) of Theorem 7.3. However, if X is finite and $\alpha, \beta \in P(X)$ have no cycles, then $s(\alpha) = s(\beta)$ does imply $\alpha \sim_c \beta$.

The transformations of a finite $P(X)$ have no rays. Hence, Theorem 7.3 gives us the following corollary.

Corollary 7.6. *Let X be finite, and let $\alpha, \beta \in P(X)$. Then $\alpha \sim_c \beta$ if and only if* $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta)$ *and $s(\alpha) = s(\beta)$.*

Example 7.7. Let α and β be partial transformations whose digraphs are presented in Figures 7.2 and 7.3, respectively. Then $\mathrm{cs}(\alpha) = \mathrm{cs}(\beta) = \{2, 3\}$ and $s(\alpha) = s(\beta) = 3$. Thus $\alpha \sim_c \beta$ by Corollary 7.6.

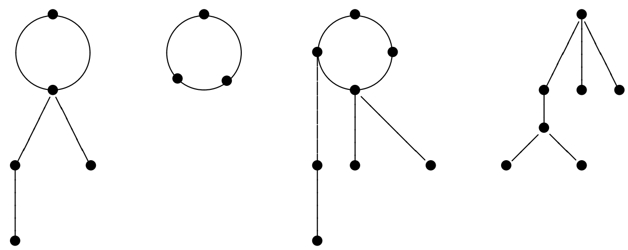

Figure 7.2: The digraph of α from Example 7.7.

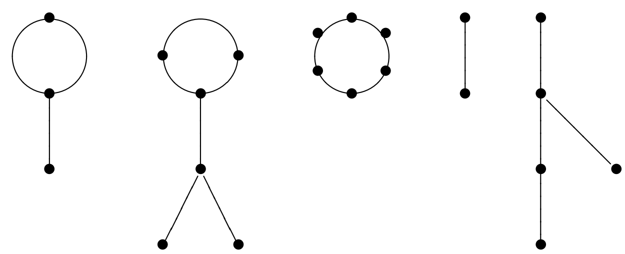

Figure 7.3: The digraph of β from Example 7.7.

We now count the conjugacy classes in an infinite $P(X)$.

Theorem 7.8. *Let X be an infinite set with $|X| = \aleph_\varepsilon$. Then in $P(X)$ there are:*

(1) $\max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\}$ *conjugacy classes containing a representative with a cycle, of which $\aleph_0$ have a connected representative;*

(2) $2^{\aleph_\varepsilon}$ *conjugacy classes containing a representative with a connected component of type rro but no cycles, of which at least $\aleph_{\varepsilon+1}$ and at most $\aleph_{\varepsilon+1}^{\aleph_0}$ have a connected representative.*

(3) $\aleph_{\varepsilon+1}$ *conjugacy classes containing a representative with a connected component of type cho but no cycles or connected components of type rro, of which $\aleph_{\varepsilon+1}$ have a connected representative.*

In total, there are $2^{\aleph_\varepsilon}$ conjugacy classes in $P(X)$, of which at least $\aleph_{\varepsilon+1}$ and at most $\aleph_{\varepsilon+1}^{\aleph_0}$ have a connected representative.

Proof. For $\alpha \in P(X)$, we define $i_\alpha, j_\alpha \in \{0, 1\}$ by $i_\alpha = 1$ if $o_\gamma(x_0) = s(\alpha)$ for some connected component of α of type *cho* with root x_0 (and $i_\alpha = 0$ otherwise); and $j_\alpha \in \{0, 1\}$ by $j_\alpha = 1$ if α has a double ray (and $j_\alpha = 0$ otherwise).

To prove (1), let $A = \{[\alpha]_c : \alpha \in P(X)$ has a cycle$\}$. Let

$$A' = \{[\alpha]_c \in A : \alpha \text{ has no maximal left rays}\} \text{ and } A'' = \{[\alpha]_c \in A : \alpha \text{ has a maximal left ray}\}.$$

By Theorem 7.3(4), $\{A', A''\}$ is a partition of A. Define $f' : A' \to \mathcal{P}(\mathbb{Z}_+) \times (\omega_{\varepsilon+1} + 1) \times \{0, 1\}$ by $([\alpha)]_c)f' = (\mathrm{cs}(\alpha), s(\alpha), i_\alpha)$. Then f' is well defined and injective by Theorem 7.3 and Lemma 6.9. (See Definition 7.5 and the discussion following the definition to see why i_α is needed.) Similarly, the mapping $f'' : A'' \to \mathcal{P}(\mathbb{Z}_+) \times (\omega_{\varepsilon+1} + 1) \times \{0, 1\}$ defined by $([\alpha)]_c)f'' = (\mathrm{cs}(\alpha), s(\alpha), i_\alpha)$ is well defined and injective. Thus

$$|A'| \leq |\mathcal{P}(\mathbb{Z}_+)| \cdot |\omega_{\varepsilon+1} + 1| \cdot 2 = 2^{\aleph_0} \cdot \aleph_{\varepsilon+1} \cdot 2 = \max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\},$$

and the same holds for $|A''|$. Hence

$$|A| = |A'| + |A''| = 2|A'| \leq 2 \max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\} = \max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\}.$$

For every nonempty set $Q \subseteq P$, where P is the set of prime positive integers, we define $\beta_Q \in P(X)$ as in the proof of Theorem 6.17. For all nonempty subsets $Q_1, Q_2 \subseteq P$ with $Q_1 \neq Q_2$, we have $(\beta_{Q_1}, \beta_{Q_2}) \notin \sim_c$ by Theorem 7.3(1). It follows that $|A| \geq \mathcal{P}(P) = 2^{\aleph_0}$. By Lemma 6.10, for every nonzero ordinal $\mu < \omega_{\varepsilon+1}$, there is $\gamma_\mu \in P(X)$ of type *cho* with root x_0 such that $o(x_0) = \mu$. For all nonzero ordinals $\lambda, \mu < \omega_{\varepsilon+1}$ with $\lambda \neq \mu$, we have $(\gamma_\lambda, \gamma_\mu) \notin \sim_c$ by Theorem 7.3(5). It follows that $|A| \geq |\omega_{\varepsilon+1}| = \aleph_{\varepsilon+1}$. Hence $|A| \geq \max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\}$, and so $|A| = \max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\}$.

Let $A_1 = \{[\gamma]_c : \gamma \in P(X)$ has a cycle and γ is connected$\}$. Fix a subset $X_0 = \{x_0, x_1, x_2, \ldots\}$ of X, and for every integer $n \geq 0$, define a connected $\gamma_n = (x_0\, x_1 \ldots x_{n-1}) \in P(X)$. Then, by Proposition 5.8 and Theorem 7.3, $A_1 = \{[\gamma_0]_c, [\gamma_1]_c, [\gamma_2]_c, \ldots\}$, and so $|A_1| = \aleph_0$. We have proved (1).

To prove (2), let

$$B = \{[\alpha]_c : \alpha \in P(X) \text{ has a connected component of type } rro \text{ but no cycles}\},$$

and let B_1 be the subset of B consisting of all conjugacy classes $[\gamma]_c \in B$ such that γ is connected. Fix a double ray $\omega = \langle \ldots x_{-1}\, x_0\, x_1 \ldots \rangle \in P(X)$ and note that

$$B_1 = \{[\gamma]_c : \gamma \in P(X) \text{ is of type } rro\} \cup \{[\omega]_c\}.$$

Let $B_1' = \{[\gamma]_c : \gamma \in P(X)$ is of type *rro*$\}$. We now follow the second paragraph of the proof of Theorem 6.17 (almost verbatim, replacing B_1 with B_1') to show that $\aleph_{\varepsilon+1} \leq |B_1'| \leq \aleph_{\varepsilon+1}^{\aleph_0}$. Then $\aleph_{\varepsilon+1} \leq |B_1| \leq \aleph_{\varepsilon+1}^{\aleph_0}$ since $|B_1| = |B_1'| + 1$.

As to the cardinality of B, clearly $|B| \leq |P(X)| = (\aleph_\varepsilon + 1)^{\aleph_\varepsilon} = 2^{\aleph_\varepsilon}$. Let

$$B' = \{[\alpha]_c \in B : \alpha \text{ has no maximal left rays or double rays}\},$$
$$B'' = \{[\alpha]_c \in B : \alpha \text{ has a maximal left ray but no double rays}\}.$$

By Theorem 7.3(3)(4), $\{B', B'', \{[\omega]_c\}\}$ is a partition of B. Next we follow the third paragraph of the proof of Theorem 6.17 (again almost verbatim, replacing B with B') to show that $|B'| \geq 2^{\aleph_\varepsilon}$. (Note that it is not necessary to define α_K as in the proof of Theorem 6.17 since we can work with β_K instead.) Finally, we follow the third paragraph of the proof of Theorem 6.17 again (this time replacing B with B'') to show that $|B''| \geq 2^{\aleph_\varepsilon}$. Hence

$$|B| = |B'| + |B''| + |\{[\omega]_c\}| \geq 2^{\aleph_\varepsilon} + 2^{\aleph_\varepsilon} + 1 = 2^{\aleph_\varepsilon},$$

and so $|B| = 2^{\aleph_\varepsilon}$. We have proved (2).

To prove (3), let C be the set of all $[\alpha]_c$ such that $\alpha \in P(X)$ has a connected component of type *cho* but no cycles or connected components of type *rro*. Let $C' = \{[\alpha]_c \in C : \alpha$ has no maximal left rays$\}$ and $C'' = \{[\alpha]_c \in C : \alpha$ has a maximal left ray$\}$. By Theorem 7.3(4), $\{C', C''\}$ is a partition of C. Fix a maximal left ray $\lambda = \langle \ldots x_2\, x_1\, x_0] \in P(X)$ and note that $C'' = \{[\lambda]_c\}$. Define $h : C' \to (\omega_{\varepsilon+1} + 1) \times \{0, 1\} \times \{0, 1\}$ by $([\alpha)]_c)h = (s(\alpha), i_\alpha, j_\alpha)$. Then

h is well defined and injective by Theorem 7.3 and Lemma 6.9, and so $|C'| \leq \aleph_{\varepsilon+1} \cdot 2 \cdot 2 = \aleph_{\varepsilon+1}$. Thus $|C| = |C'| + |C''| = |C'| + 1 \leq \aleph_{\varepsilon+1} + 1 = \aleph_{\varepsilon+1}$.

Let C_1 be the subset of C consisting of all $[\gamma]_c \in C$ such that γ is connected. Note that $C_1 = \{[\gamma]_c : \gamma \in P(X) \text{ is of type } cho\} \cup \{[\lambda]_c\}$. As in the proof of (1), we can construct a collection $\{\gamma_\mu\}_{0<\mu<\omega_{\varepsilon+1}}$ of connected elements of $P(X)$ of type *cho* such that $(\gamma_\lambda, \gamma_\mu) \notin \sim_c$ if $\lambda \neq \mu$. Thus $|C_1| \geq \aleph_{\varepsilon+1}$, and so $\aleph_{\varepsilon+1} \leq |C_1| \leq |C| \leq \aleph_{\varepsilon+1}$. Hence $|C| = |C_1| = \aleph_{\varepsilon+1}$, which concludes the proof of (3).

The conjugacy classes considered in (1)–(3) cover all conjugacy classes in $P(X)$. Thus, there are at most $\max\{2^{\aleph_0}, \aleph_{\varepsilon+1}\} + 2^{\aleph_\varepsilon} + \aleph_{\varepsilon+1} = 2^{\aleph_\varepsilon}$ conjugacy classes in $P(X)$ (which also follows from the fact that $|P(X)| = 2^{\aleph_\varepsilon}$). By (2), there are at least $2^{\aleph_\varepsilon}$ conjugacy classes, so the number of conjugacy classes in $P(X)$ is $2^{\aleph_\varepsilon}$. By (1)–(3), at least $\aleph_{\varepsilon+1}$ and at most $\aleph_0 + \aleph_{\varepsilon+1}^{\aleph_0} + \aleph_{\varepsilon+1} = \aleph_{\varepsilon+1}^{\aleph_0}$ of these conjugacy classes have a connected representative. □

8 Conjugacy in $\Gamma(X)$

By $\Gamma(X)$ we mean the subsemigroup of $T(X)$ consisting of injective transformations. If X is finite, then $\Gamma(X) = \mathrm{Sym}(X)$ but this is not the case for an infinite X. The semigroup $\Gamma(X)$ is universal for right cancellative semigroups with no idempotents (except possibly the identity): that is, any such semigroup can be embedded in $\Gamma(X)$ for some X [17, Lemma 1.0]. The semigroup $\Gamma(X)$ has been studied mainly in the context of: ideals and congruences [36, 53]; $\mathcal{G}(X)$-normal semigroups [34, 35, 51]; Baer-Levi semigroups [37, 38]; BQ-semigroups [28, 52], and centralizers [29, 30]. In this section, we characterize the conjugacy $\sim_c$ in $\Gamma(X)$ for an arbitrary set X.

We note that every connected transformation in $P(X)$ that is also injective is a cycle, a ray, or a chain. Since transformations in $\Gamma(X)$ are full, $\alpha \in \Gamma(X)$ cannot contain a maximal left ray or a maximal chain. These observations give the following lemma.

Lemma 8.1. *Let $\alpha \in \Gamma(X)$. Then every connected component of α is a right ray, a double ray, or a cycle.*

The following proposition follows from Lemma 8.1 and Proposition 5.5.

Proposition 8.2. *Let $\alpha \in \Gamma(X)$. Then there exist unique sets: A of right rays, B of double rays, and C of cycles such that the transformations in $A \cup B \cup C$ are pairwise completely disjoint and*

$$\alpha = \left(\bigsqcup_{\eta \in A} \eta \right) \sqcup \left(\bigsqcup_{\omega \in B} \omega \right) \sqcup \left(\bigsqcup_{\lambda \in C} \lambda \right).$$

Let $\alpha \in \Gamma(X)$. We will denote the unique sets A, B, and C from Proposition 8.2 by A_α, B_α, and C_α, respectively. For $n \geq 1$, we will denote by C_α^n the subset of C_α consisting of cycles of length n. Note that:

$$\begin{aligned} A_\alpha &= \text{the set of maximal right rays contained in } \alpha, \\ B_\alpha &= \text{the set of double rays contained in } \alpha, \\ C_\alpha &= \text{the set of cycles contained in } \alpha. \end{aligned}$$

For $\eta = (x_0\, x_1\, x_2 \ldots\rangle$, $\omega = \langle \ldots x_{-1}\, x_0\, x_1 \ldots\rangle$, $\lambda = (x_0\, x_1 \ldots x_{k-1})$, and any ϕ in $\Gamma(X)$, we define:

$$\eta\phi^* = (x_0\phi\, x_1\phi\, x_2\phi \ldots\rangle,\ \omega\phi^* = \langle \ldots x_{-1}\phi\, x_0\phi\, x_1\phi \ldots\rangle,\ \lambda\phi^* = (x_0\phi\, x_1\phi \ldots x_{k-1}\phi).$$

Proposition 8.3. *Let $\alpha, \beta, \phi \in \Gamma(X)$. Then ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$ if and only if for all $\eta \in A_\alpha$, $\omega \in B_\alpha$, and $\lambda \in C_\alpha$:*

(1) *Either there is a unique* $\eta_1 \in A_\beta$ *such that* $\eta\phi^* \sqsubset \eta_1$ *or there is a unique* $\omega_1 \in B_\beta$ *such that* $\eta\phi^* \sqsubset \omega_1$*;*

(2) $\omega\phi^* \in B_\beta$ *and* $\lambda\phi^* \in C_\beta$*.*

Proof. Suppose ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Let $\eta = (x_0\, x_1\, x_2 \ldots\rangle \in A_\alpha$. Then, since ϕ is an injective homomorphism, $\eta\phi^* = (x_0\phi\, x_1\phi\, x_2\phi \ldots\rangle$ is a right ray in $\Gamma(\beta)$. By the proof of Proposition 6.1, $\phi|_{\mathrm{span}(\eta)}$ is a homomorphism from $\Gamma(\eta)$ to $\Gamma(\gamma)$ for some connected componnet γ of β. By Lemma 8.1, either $\gamma = \eta_1 = (y_0\, y_1\, y_2 \ldots\rangle$ is a right ray in β or $\gamma = \omega_1 = \langle \ldots y_{-1}\, y_0\, y_1 \ldots\rangle$ is a double ray in β (γ cannot be a cycle since ϕ is injective). In the former case, $\eta\phi^* \sqsubset \eta_1$, and in the latter case, $\eta\phi^* \sqsubset \omega_1$. The uniqueness of η_1 and ω_1 follows from the fact that the elements of $A_\beta \cup B_\beta$ are pairwise completely disjoint. We have proved (1). The proof of (2) is similar.

Conversely, suppose that ϕ satisfies (1) and (2). Then it follows immediately that for all $x, y \in X$, $x \xrightarrow{\alpha} y$ implies $x\phi \xrightarrow{\beta} y\phi$, and so ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. ◻

Definition 8.4. Let $\alpha, \beta \in \Gamma(X)$. For a homomorphism $\phi \in \Gamma(X)$ from $\Gamma(\alpha)$ to $\Gamma(\beta)$, we define a mapping $h_\phi : A_\alpha \cup B_\alpha \cup C_\alpha \to A_\beta \cup B_\beta \cup C_\beta$ by:

$$\delta h_\phi = \begin{cases} \eta & \text{if } \delta \in A_\alpha \text{ and } \delta\phi^* \sqsubset \eta \text{ for some } \eta \in A_\beta, \\ \omega & \text{if } \delta \in A_\alpha \text{ and } \delta\phi^* \sqsubset \omega \text{ for some } \omega \in B_\beta, \\ \delta\phi^* & \text{if } \delta \in A_\alpha \cup C_\alpha. \end{cases}$$

Note that h_ϕ is well defined (by Proposition 8.3) and injective (since ϕ is injective).

We will need the following lemma from set theory (whose proof is straightforward).

Lemma 8.5. *Let* A_1*,* B_1*,* A_2*, and* B_2 *be sets such that* $A_1 \cap B_1 = \emptyset$*,* $A_2 \cap B_2 = \emptyset$*,* $|A_1| + |B_1| \leq |A_2| + |B_2|$*, and* $|B_1| \leq |B_2|$*. Then there is an injective mapping* $f : A_1 \cup B_1 \to A_2 \cup B_2$ *such that* $xf \in B_2$ *for every* $x \in B_1$*.*

We can now characterize the conjugacy $\sim_c$ in $\Gamma(X)$.

Theorem 8.6. *Let* $\alpha, \beta \in \Gamma(X)$*. Then* $\alpha \sim_c \beta$ *in* $\Gamma(X)$ *if and only if* $|A_\alpha| + |B_\alpha| = |A_\beta| + |B_\beta|$*,* $|B_\alpha| = |B_\beta|$*, and* $|C^n_\alpha| = |C^n_\beta|$ *for every* $n \geq 1$*.*

Proof. Suppose $\alpha \sim_c \beta$ in $\Gamma(X)$. Then, by Corollary 4.8, there is $\phi \in \Gamma(X)$ such that ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. Define $f : A_\alpha \cup B_\alpha \to A_\beta \cup B_\beta$ by $\delta f = \delta h_\phi$. (By the definitions of h_ϕ and ϕ^*, δf is indeed in $A_\beta \cup B_\beta$ if $\delta \in A_\alpha \cup B_\alpha$.) The mapping f is injective (since h_ϕ is injective), $A_\alpha \cap B_\alpha = \emptyset$, and $A_\beta \cap B_\beta = \emptyset$. Thus

$$|A_\alpha| + B_\alpha| = |A_\alpha \cup B_\alpha| \leq |A_\beta \cup B_\beta| = |A_\beta| + |B_\beta|.$$

Similarly, $|B_\alpha| \leq |B_\beta|$ since $g : B_\alpha \to B_\beta$ defined by $\omega g = \omega h_\phi$ is well defined and injective. Let $n \geq 1$. Define $h : C^n_\alpha \to C^n_\beta$ by $\lambda h = \lambda h_\phi$. (If $\lambda = (x_0 \ldots x_{n-1}) \in C^n_\alpha$, then $\lambda h_\phi = \lambda\phi^* = (x_0\phi \ldots x_{n-1}\phi) \in C^n_\beta$.) The mapping h is injective, and so $|C^n_\alpha| \leq |C^n_\beta|$. By symmetry, $|A_\beta| + |B_\beta| \leq |A_\alpha| + |B_\alpha|$, $|B_\beta| \leq |B_\alpha|$, and $|C^n_\beta| \leq |C^n_\alpha|$. Hence the stated equalities hold.

Conversely, suppose $|A_\alpha| + |B_\alpha| = |A_\beta| + |B_\beta|$, $|B_\alpha| = |B_\beta|$, and $|C^n_\alpha| = C^n_\beta|$ for every $n \geq 1$. We will define an injective homomorphism ϕ from $\Gamma(\alpha)$ to $\Gamma(\beta)$. By Lemma 8.5, there is an injective mapping $f : A_\alpha \cup B_\alpha \to A_\beta \cup B_\beta$ such that $\omega f \in B_\beta$ for every $\omega \in B_\alpha$. For every $n \geq 1$, fix a bijection $g_n : C^n_\alpha \to C^n_\beta$. Let $n \geq 1$. For all $\eta \in A_\alpha$, $\omega \in B_\alpha$, and $\lambda \in C^n_\alpha$, we define ϕ on $\mathrm{dom}(\eta) \cup \mathrm{dom}(\omega) \cup \mathrm{dom}(\lambda)$ in such a way that $\eta\phi^* \sqsubset \eta f$, $\omega\phi^* = \omega f$, and $\lambda\phi^* = \lambda g_n$. Note that this defines ϕ for every $x \in X$. By the definition of ϕ and Proposition 8.3, $\phi \in \Gamma(X)$ and ϕ is a homomorphism from $\Gamma(\alpha)$ to $\Gamma(\beta)$. By symmetry, there is an injective homomorphism ψ from $\Gamma(\beta)$ to $\Gamma(\alpha)$. Hence $\alpha \sim_c \beta$ by Corollary 4.8. ◻

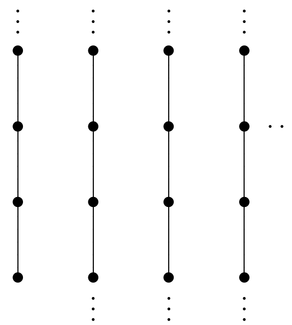

Figure 8.1: The digraph of α from Example 8.7.

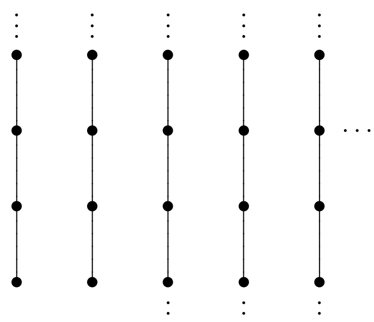

Figure 8.2: The digraph of β from Example 8.7.

Example 8.7. Let α and β be partial transformations on an infinite set whose digraphs are presented in Figures 8.1 and 8.2, respectively. Then $|A_\alpha| = 1$, $|B_\alpha| = \aleph_0$, $|A_\alpha| + |B_\alpha| = \aleph_0$, and $|C^n_\alpha| = 0$ for every $n \geq 1$. Also, $|A_\beta| = 2$, $|B_\beta| = \aleph_0$, $|A_\beta| + |B_\beta| = \aleph_0$, and $|C^n_\beta| = 0$ for every $n \geq 1$. Thus $\alpha \sim_c \beta$ by Theorem 8.6.

Using Theorem 8.6, we can count the conjugacy classes in $\Gamma(X)$. First, we need the following lemma.

Lemma 8.8. *Let X be an infinite set with $|X| = \aleph_\varepsilon$, let $\alpha \in \Gamma(X)$. Then $|A_\alpha| \leq \aleph_\varepsilon$, $|B_\alpha| \leq \aleph_\varepsilon$, and $|C^n_\alpha| \leq \aleph_\varepsilon$ for every $n \geq 1$.*

Proof. Let $Y = \bigcup_{\eta \in A_\alpha} \mathrm{dom}(\eta) \subseteq X$. Since the elements of A_α are pairwise completely disjoint and $|\,\mathrm{dom}(\eta)| = \aleph_0$ for every $\eta \in A_\alpha$, we have

$$\aleph_\varepsilon = |X| \geq |Y| = |\bigcup_{\eta \in A_\alpha} \mathrm{dom}(\eta)| = |A_\alpha| \cdot \aleph_0 \geq |A_\alpha|.$$

Thus $A_\alpha| \leq \aleph_\varepsilon$. The proofs for B_α and C^n_α $(n \geq 1)$ are similar. □

For sets A and B, we denote by A^B the set of all functions from B to A.

Theorem 8.9. *Let X be an infinite set with $|X| = \aleph_\varepsilon$. Let $\kappa = \aleph_0 + |\varepsilon|$. Then there are $\kappa^{\aleph_0}$ conjugacy classes in $\Gamma(X)$, of which two have a connected representative if $\aleph_\varepsilon = \aleph_0$, and none has a connected representative if $\aleph_\varepsilon > \aleph_0$.*

Proof. Let K be the set of all cardinals τ such that $\tau \leq \aleph_\varepsilon$. Then K contains $\aleph_0$ finite cardinals and $|\varepsilon| + 1$ infinite cardinals, hence $|K| = \aleph_0 + |\varepsilon| + 1 = \aleph_0 + |\varepsilon| = \kappa$. Let $\Gamma(X)/\sim_c$ be the set of

conjugacy classes of $\Gamma(X)$. Define a function $f : \Gamma(X)/{\sim_c} \to K^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, by

$$([\alpha)]_c)f = (|A_\alpha| + |B_\alpha|, |B_\alpha|, |C^1_\alpha|, |C^2_\alpha|, |C^3_\alpha|, \ldots).$$

By Theorem 8.6, f is well defined and injective. Thus $|\Gamma(X)/{\sim_c}| \leq |K^{\mathbb{N}}| = |K|^{|\mathbb{N}|} = \kappa^{\aleph_0}$.

We next define an injective mapping $g : K^{\mathbb{N}} \to \Gamma(X)/{\sim_c}$. Let

$$\xi = (\tau_2, \tau_3, \tau_4, \ldots) \in K^{\mathbb{N}}.$$

(It will be clear from the definition of g why we begin the indexing with $n = 2$.) Let $\tau = \sum_{n=2}^{\infty} n\tau_n$ (see [27, Chapter 9]). For every $n \geq 2$, $n\tau_n \leq \aleph_\varepsilon$ (since $\tau_n \leq \aleph_\varepsilon$ and $\aleph_\varepsilon$ is infinite). Thus

$$\tau = \sum_{n=2}^{\infty} n\tau_n \leq \aleph_0 \cdot \aleph_\varepsilon = \aleph_\varepsilon,$$

and so $\aleph_\varepsilon + \tau = \aleph_\varepsilon$. Hence, there is a collection $\{X_n\}_{n\geq 1}$ of pairwise disjoint subsets of X such that $\bigcup_{n=1}^{\infty} X_n = X$, $|X_1| = \aleph_\varepsilon$, and $|X_n| = n\tau_n$ for every $n \geq 2$. Let $n \geq 2$. Since $|X_n| = n\tau_n$, there is a collection C_n of n-cycles in $\Gamma(X)$ such that $|C_n| = \tau_n$ and $\mathrm{dom}(\bigsqcup_{\lambda\in C_n} \lambda) = X_n$. Let $\alpha_n = \bigsqcup_{\lambda\in C_n} \lambda$. Define a transformation α_ξ on X by

$$\alpha_\xi = \bigsqcup_{n\geq 2} \alpha_n \sqcup \bigsqcup_{x\in X_1} (x).$$

Then $\alpha \in \Gamma(X)$, $A_\alpha = B_\alpha = \emptyset$, and $C^n_\alpha = C_n$ for all $n \geq 2$. Thus

$$(|C^1_\alpha|, |C^2_\alpha|, |C^3_\alpha|, |C^4_\alpha|, \ldots) = (\aleph_\varepsilon, \tau_2, \tau_3, \tau_4, \ldots),$$

and it follows from Theorem 8.6 that the mapping $g : K^{\mathbb{N}} \to \Gamma(X)/{\sim_c}$ defined by $\xi g = \alpha_\xi$ is injective. Hence $|\Gamma(X)/{\sim_c}| \geq |K^{\mathbb{N}}| = |K|^{|\mathbb{N}|} = \kappa^{\aleph_0}$.

Suppose $|X| = \aleph_0$, say $X = \{x_1, x_2, x_3, \ldots\}$. Then, by Theorem 8.6 and Lemma 8.1, the only conjugacy classes in $\Gamma(X)$ with a connected representative are $[(x_1\, x_2\, x_3 \ldots\rangle]$ and $[\langle \ldots x_6\, x_4\, x_2\, x_1\, x_3\, x_5 \ldots\rangle]$. (There is no single cycle in λ in $\Gamma(X)$ since $\mathrm{dom}(\lambda)$ is finite.)

If $|X| > \aleph_0$, then no element $\alpha \in \Gamma(X)$ is connected since $\mathrm{dom}(\alpha) = X$ and the domain of any right ray, double ray, or cycle has cardinality at most $\aleph_0$. The result follows. $\square$

9 Problems

The results of this paper prompt a number of problems on combinatorics, semigroups, matrix theory, and set theory. The first problem asks for the number of conjugacy classes in some important finite semigroups.

Problem 9.1. Let X be a finite set. Is it possible to find a closed formula that gives the number of conjugacy classes in $T(X)$, $P(X)$ or $\mathcal{I}(X)$ (where $\mathcal{I}(X)$ denotes the symmetric inverse semigroup on X)?

The second problem might attract the attention of experts in set theory.

Problem 9.2. Let X be an infinite set with $|X| = \aleph_\varepsilon$. According to Theorem 6.17, the number of conjugacy classes in $T(X)$ that have a connected representative is in the interval $[\aleph_{\varepsilon+1}, \aleph_{\varepsilon+1}^{\aleph_0}]$. Is it possible to be more precise and reduce the length of this interval?

In this paper we characterized the conjugate elements in some well-known transformation semigroups, but there are many other transformation semigroups, or endomorphism monoids of some relational algebras that may be considered.

Problem 9.3. Characterize $\sim_c$, and calculate the number of conjugacy classes, in other transformation semigroups such as, for example, those appearing in the problem list of [7, Section 6] or those appearing in the large list of transformation semigroups included in [20]. Especially interesting would be a characterization of the conjugacy classes in the centralizers of idempotents [6, 5].

The theorems and problems in this paper have natural linear counter-parts.

Problem 9.4. Characterize $\sim_c$ in the endomorphism monoid of a (finite or infinite dimensional) vector space.

Whenever some result holds for both sets and vector spaces the natural step forward is to prove those results for independence algebras.

Problem 9.5. Characterize $\sim_c$ in the endomorphism monoid of a (finite or infinite dimensional) independence algebra. (For historical notes on the importance of these algebras, see [4, 3]; for definitions and basic results, see [1, 2, 8, 9, 14, 21, 22, 23]).

Problem 9.6. The notion of conjugation $\sim_p$ defined in (1.2) is very important in symbolic dynamics in connection with the Williams Conjecture [54]. Characterize $\sim_p$ in $T(X)$, $P(X)$ and $\mathcal{I}(X)$ for an infinite set X. (Kudryavtseva and Mazorchuk [31] have characterized $\sim_p^*$ (the trensitive closure of $\sim_p$) in $T(X)$, $P(X)$ and $\mathcal{I}(X)$ for a finite X, and in $\mathcal{I}(X)$ in a countable X.)

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